A METEOROLOGICAL TREATISE

ON THE

Circulation and Radiation

IN THE ATMOSPHERES OF THE EARTH AND OF THE SUN

BY

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INTRODUCTION

METEOROLOGY as a science has failed to make progress toward definite results for one fundamental reason. In a non-adiabatic atmosphere the terms of the general equations of motion, as computed from the ordinary prescribed formulas of thermodynamics, do not balance as required. There are two errors in the discussion: (1) There is a mixture of the non-adiabatic and the adiabatic systems, and (2) the important radiation terms have been omitted from the general equations. More specifically, for the Boyle-Gay Lussac Law, $P = \rho T R$, to be satisfied, at every point, it has been customary to borrow R = gas constant from the adiabatic system, and apply it in the non-adiabatic atmosphere. For example, three well known treatments follow:

Bigelow.

$$\frac{P_1}{P_0} = \left(\frac{T_1}{T_0}\right)^{\frac{nk}{k-1}} \qquad \frac{\rho_1}{\rho_0} = \left(\frac{T_1}{T_0}\right)^{\frac{n}{k-1}} \qquad \frac{R_1}{R_0} = \left(\frac{T_1}{T_0}\right)^{n-1}$$

v. Bjerknes.

$$\frac{P_1}{P_0} = \left(\frac{T_1}{T_0}\right)^{\frac{nk}{k-1}} \qquad \frac{\rho_1}{\rho_0} = \left(\frac{T_1}{T_0}\right)^{\frac{n}{k-1}+(n-1)} \qquad R_1 = R_0 = \text{Constant}$$

Margules.

$$\frac{P_1}{P_0} = \left(\frac{T_1}{T_u}\right)^{\frac{k}{k-1}} \qquad \frac{\rho_1}{\rho_0} = \left(\frac{T_1}{T_u}\right)^{\frac{1}{k-1}} \qquad R_1 = R_0 = \text{Con.}$$

Margules' system is adiabatic, v. Bjerknes' is partly adiabatic and partly non-adiabatic, Bigelow's is strictly non-adiabatic.

$$n = \frac{a_0}{a} = \frac{T_a - T_0}{T_1 - T_0},\ k = \frac{Cp.}{Cu}$$

Now it is true that each system satisfies $P = \rho T R$, but the individual values of P_1, T_1, ρ_1, R_1, are very different in the three systems for the same initial values P_0, T_0, ρ_0, R_0, and on applying them to practical observations the systems that are not strictly non-adiabatic break down as regards computed and observed values which should be in agreement. We can easily see the separate consequences by the following equations that are readily demonstrated:

Adiabatic: $g(z_1 - z_0) = - Cp_a(T_a - T_0) = - \frac{P_a - P_0}{\rho_{a0}}$.

$$g(z_1 - z_0) = - n_1 Cp_a (T_1 - T_0) = - \frac{P_a - P_0}{\rho_{a0}}.$$

Non-adiabatic: $g(z_1 - z_0) = - n_1 Cp_{10} (T_1 - T_0) - n_1 (Cp_a - Cp_{10}) (T_1 - T_0)$.

$$g(z_1 - z_0) = - \frac{P_1 - P_0}{\rho_{10}} - \tfrac{1}{2}(q_1^2 - q_0^2) - (Q_1 - Q_0).$$

Gravity term. = Pressure term. + Circulation term. + Radiation term.

In the adiabatic system by definition R_a and Cp_a are constant, and there are no circulation and no radiation. In the non-adiabatic system there are both circulation and radiation, but these depend upon the departure of the specific heat from the adiabatic value

$$n_1 (Cp_a - Cp_{10}) (T_1 - T_0) = \tfrac{1}{2}(q^2_1 - q^2_0) + (Q_1 - Q_0).$$

Hence, if R is constant it is a contradiction in terms to discuss problems of circulation and radiation in the adiabatic or partially adiabatic systems, as has been universally the method.

The facts of observation, furthermore, conform to

$$- n_1 Cp_{10} (T_1 - T_0) = - \frac{P_1 - P_0}{\rho_{10}},$$

and this implies that circulation and radiation are required to make up the deficiency in respect to the gravity term between two strata z_1 and z_0. Finally, the radiation term $(Q_1 - Q_0)$ is usually very much larger than the kinetic energy term $\tfrac{1}{2}(q_1^2 - q_0^2)$, but it has never been incorporated in the meteorological equations.

For these reasons the author has spent much time while in the United States Weather Bureau, and especially while in the

Argentine Meteorological Office, 1910 to date, in devising a simple adjustment of the thermodynamic adiabatic equations, found in all treatises, to an exact and practical form of computation which will adapt them to the non-adiabatic system prevailing in the atmospheres of the earth and of the sun. The following Treatise sets forth this new method of discussing the meteorological problems, with sufficient detail to enable the reader to utilize the formulas in practical computations. It contains the solution of a number of problems that have heretofore been intractable along the old lines of procedure:

1. The diurnal convection and the semi-diurnal barometric waves, with the radiation.

2. The pressures and temperatures in cyclones and anticyclones, with the circulation and radiation.

3. The thermodynamics of the atmosphere from balloon ascensions to great altitudes.

4. The thermodynamics of the general circulation.

5. The distribution of the radiation in all latitudes and altitudes to 20,000 meters.

6. The "solar constant" of radiation and the conflicting results from pyrheliometers and bolometers.

7. The discrepancy in the absolute coefficient of electrical conduction as derived from the several apparatus for dissipation, and for the number and velocity of the ions.

8. The diurnal magnetic variations in the lower strata of the atmosphere.

9. The non-periodic magnetic variations in their relation to the solar radiation.

10. The magnetization and electrical terms in the sun at very high temperatures.

I wish to express my appreciation and gratitude to my friend and colleague, Dr. Walter G. Davis, Director of the Argentine Meteorological Office, for his courteous co-operation in this work, and in memory of our good old Cordoba days.

FRANK H. BIGELOW.

CORDOBA, ARGENTINA.
September, 1915.

TABLE OF CONTENTS

CHAPTER I

CHAPTER II

CHAPTER III

CHAPTER IV

CHAPTER V

CHAPTER VI

CHAPTER VII

A METEOROLOGICAL TREATISE ON THE CIRCULATION AND RADIATION IN THE ATMOSPHERES OF THE EARTH AND OF THE SUN

CHAPTER I

Meteorological Constants and Elementary Formulæ

The Status of Meteorology

Modern meteorology may be said to have begun its scientific development about the year 1870. Since that time an enormous mass of observations has been made, covering every branch of the subject, but the classification of the data in any systematic form has been singularly inadequate. This defective progress may be attributed chiefly to two causes, the first, the practical units in which the instrumental readings are commonly made, and the second, the usual misapplication of the Boyle-Gay Lussac Law in the atmosphere, whereby it is assumed that the gas coefficient is constant and the conditions strictly adiabatic. Thus, for the pressure P_0, the density ρ_0, the temperature T_0, and the gas coefficient R_0, the law is,

(1) $$P_0 = \rho_0 R_0 T_0,$$

and it must be satisfied at every point, but in passing from one point to another, whereby $P_1 \rho_1 T_1$ change values, the observations made in balloon and kite ascensions are not verified without making R change with the height above the sea-level. This variation of R carries with it a variability of the specific heat by the formula,

(2) $$Cp = \frac{k}{k-1} R,$$

where the ratio of the specific heats is $k = Cp/Cv$, and thence the changes from one level to another follow, for the free heat $(Q_1 - Q_0)$, the entropy $(S_1 - S_0)$, the work $(W_1 - W_0)$, the inner energy $(U_1 - U_0)$, the radiation energy $(K_1 - K_0)$, and the coefficient c and exponent a in the radiation function,

$$K = c\,T^a. \tag{3}$$

It can be proved, furthermore, that unless Cp varies from the adiabatic specific heat Cp_a for R_a constant, there can be no circulation and no radiation of heat in the atmosphere, such as actually exists. It is the purpose of this Treatise to develop the working formulæ on a systematic plan, so that they shall be adapted to the computation of the necessary data throughout the atmosphere, as in the diurnal convection, the local and cyclonic circulation, and in the general planetary circulation, by means of which numerous problems may be studied without an undue amount of speculation.

The difficulty regarding the type of the observations available is that the records are not suitable for use in the formulæ without special reductions, and that a number of different sets of units are employed. There are three systems of units in more or less consistent use, (1) the meter-kilogram-second-Centigrade degree system (M. K. S. C°), (2) the centimeter-gram-second-Centigrade degree system (C. G. S. C°), (3) the foot-pound-second-Fahrenheit degree system (F. P. S. F°). The pressure, B_n, for the fundamental formulæ is recorded in barometric millimeters of mercury, or in inches of mercury, instead of in force units, employing the acceleration of gravity,

$$g_{\phi z} = 9.8060\,(1 - 0.00260 \cos 2\phi)\left(1 - \frac{2z}{R}\right), \tag{4}$$

where 9.8060 in meters is the acceleration per second in latitude $\varphi = 45°$, and at sea-level $z = 0$, for $R = 6370191$ meters, the radius of the earth. The pressure of one atmosphere in units of force is,

$$P_0 = g_0\,\rho_m\,B_0, \tag{5}$$

where $\rho_m = 13595.8$ kilograms per cubic meter, the density of mercury. Similarly,

$$P_0 = g_0\,\rho_0\,l_0, \tag{6}$$

where $\rho_0 = 1.29305$ density of air in the same units, and $l_0 =$

7991.04 the height of the homogeneous atmosphere. For B_n = 0.760 meter, the pressure P_0 = 101323.5 kilograms per square meter. Again, in units of mass,

(7) $$p_0 = \frac{P_0}{g_0} = \rho_m B_0 = \rho_0 l_0 = 10332.8.$$

Finally, in heat units,

(8) $$p_A = \frac{P_0}{g_0 \, 426.837} = 24.2106$$

where the mechanical equivalent of heat is

(9) A = 426.837 kilogram meters, one large calorie.

The pressure of the atmosphere is, therefore,

P_0 = 101323.5 in units of force,
p_0 = 10332.8 in units of mass,
p_A = 24.2106 in units of heat,
B_0 = 0.760 in barometric units,

so that the common barometer readings must be wholly transformed for practical computations in dynamic and thermodynamic problems which involve circulation and radiation.

Similarly the (C. G. S. C°) system and the (F. P. S. F°) system each involves four other values for one atmospheric pressure, and it often happens that there occur crosses between these three systems, so that meteorological literature is confused and difficult to comprehend in any clear manner.

The temperature is usually recorded in Centigrade degrees, C°, or in Fahrenheit degrees, F°, but the computations must be executed in degrees of absolute temperature.

(10) $T = 273° + t$ (C°) Centigrade,

(11) $T = 459.°4 + t$ (F°) Fahrenheit,

so that another series of transformations is required. It would improve matters greatly to mark barometers in terms of P_0 and thermometers in degrees T, and reconstruct the entire series of working tables. The relations between the barometric and the force pressures are as follows:

TABLE 1

1 Scale division = 0.75 mm. ΔB = 100. ΔP

ΔB in millimeters	ΔP units of force	Unit Equivalents
100.	13332.	
10.	1333.2	1.0 mm. ΔB = 133.32 ΔP
1.	133.32	
0.1	13.332	0.75 mm. ΔB = 100. ΔP
75.	10000.	
7.5	1000.	0.0075 mm. ΔB = 1.0 ΔP
0.75	100.	
0.075	10.	

If the divisions on the barometer scale are made 0.75 millimeter, the scale distance is 100 units of force in the (M. K. S. C°) system, so that the readings of the barometric pressure are immediately available for all forms of dynamic and thermodynamic meteorology. This scale is equally valuable for public purposes and synoptic chart construction.

If the absolute temperatures (T_0, T_1) are observed on two levels of the atmosphere at the heights (z_0, z_1), respectively, the formulas permit the corresponding values to be computed for (P_0, P_1), (ρ_0, ρ_1), (R_0, R_1), and (Cp_0, Cp_1). It can be shown that

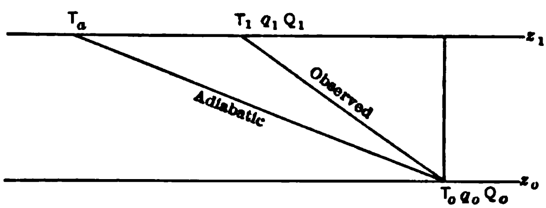

FIG. 1. The temperature, velocity, and heat in the stratum ($z_1 - z_0$)

if Cp_a is the specific heat for the adiabatic R_a, and Cp_{10} is the mean specific heat for the stratum (z_0, z_1), the circulation and the radiation are given by the formula,

$$(12) \quad (Cp_a - Cp_{10})\,(T_a - T_0) = \tfrac{1}{2}\,(q_1^2 - q_0^2) + (Q_1 - Q_0),$$

where the terms are related to the points as indicated in Fig. 1 for the stratum (z_0, z_1).

If $(T_a - T_0)$ is the adiabatic temperature fall in the vertical distance $(z_1 - z_0)$, and $(T_1 - T_0)$ the actual observed temperature fall, the ratio between them is,

$$(13) \qquad n = \frac{-(T_a - T_0)}{-(T_1 - T_0)} = \frac{a_0 (z_1 - z_0)}{a (z_1 - z_0)} = \frac{a_0}{a}.$$

Again, if (q_0, q_1) are the corresponding velocities without regard to direction, and $(Q_1 - Q_0)$ the loss of free heat in the stratum, these quantities are all connected together by the equation (12). This states that the kinetic energy of the circulation for the unit mass $\frac{1}{2}(q_1^2 - q_0^2)$, together with the heat exchange of the radiation energy of the free heat $(Q_1 - Q_0)$, depends upon the divergence of the actual mean specific heat Cp_{10} from the adiabatic value,

$$(14) \qquad Cp_a = 993.5787 \text{ at constant pressure.}$$

The corresponding values for Cv_a amd R_a are

$$(15) \qquad Cv_a = 706.5453 \text{ at constant volume.}$$

$$Cp_a - Cv_a = R_a = 287.0334 \text{ the gas coefficient.}$$

Hence, $\frac{Cp_a}{Cv_a} = k = 1.4062486$ the ratio of the specific heats.

$$(16) \qquad Cp_a - Cv_a = Cv_a (k - 1)$$

$$(17) \qquad \frac{Cp_a}{Cp_a - Cv_a} = \frac{Cp_a}{R_a} = \frac{k}{k - 1} = 3.461545.$$

$$(18) \qquad \frac{Cv_a}{Cp_a - Cv_a} = \frac{Cv_a}{R_a} = \frac{1}{k - 1} = 2.461545.$$

The corresponding values in a non-adiabatic atmosphere become,

$$(19) \qquad \frac{Cp}{Cp - Cv} = \frac{Cp}{R} = \frac{k}{k - 1}.$$

$$(20) \qquad \frac{Cv}{Cp - Cv} = \frac{Cv}{R} = \frac{1}{k - 1}.$$

where $\qquad R = R_a \left(\frac{T}{T_a}\right)^{n-1}$ as will be proved.

These two systems will be more fully developed, it being the purpose here merely to point out the leading line of the construction of this work. By the definition of the specific heat,

$$(21) \quad + Cp_a\,(T_a - T_0) = - g_0\,(z_1 - z_0),$$

for the unit mass, so that

$$(22) \quad a_0 = - \frac{T_a - T_0}{z_1 - z_0} = \frac{g_0}{Cp_a} = 0.0098695\ C°/\text{meter},$$

$$(23) \quad a = - \frac{T_1 - T_0}{z_1 - z_0} = \frac{g_0}{Cp_{10}} = \frac{g_0}{n\,Cp_a} = \frac{a_0}{n}\ C°/\text{meter}.$$

From (17), (19), (22), and (23)

$$(24) \quad \frac{n\,k}{k-1} = \frac{nCp_a}{R_a} = \frac{n\,g_0}{R_a . a_0} = \frac{g_0}{R_a}\frac{1}{a} = - \frac{g_0}{R_a} \cdot \frac{z_1 - z_0}{T_1 - T_0}.$$

It will be shown from the data of observation that

$$(25) \quad Cp_{10}\,(T_a - T_0) = \frac{P_1 - P_0}{\rho_{10}}.$$

Subtracting (25) from (21),

$$(26) \quad (Cp_a - Cp_{10})\,(T_a - T_0) = - g_0\,(z_1 - z_0) - \frac{P_1 - P_0}{\rho_{10}}.$$

Equating (12) and (26),

$$(27) \quad - g_0\,(z_1 - z_0) = + \frac{P_1 - P_0}{\rho_{10}} + \tfrac{1}{2}\,(q_1^2 - q_0^2) + (Q_1 - Q_0),$$

and this is the general equation of condition in the atmosphere, showing that for the unit mass the force of gravitation is balanced by the change of pressure, the kinetic energy of the circulation, and the radiating heat.

Now, returning to the registration of the fundamental quantities, in addition to the temperature T, and the pressure P, there is also the velocity q. While the sum of the energy of the circulation and the radiation can be computed through the difference of the specific heats $(Cp_a - Cp_{10})$, there is no way to separate the circulation from the radiation except through the direct observation of the velocity. The radiation must be computed indirectly through the gravitation, pressure, and circulation terms taken together in an inequality. As these three terms seldom balance in the free atmosphere, which is continually exchanging heat at every point, it is evident that the adiabatic conditions, in connection with the general equation of motion, are not capable of giving a complete solution of any of the important thermodynamic problems of the atmosphere. The literature of meteorology is defective in this respect. It should be noted that the point of departure for this treatment of the problem consists in making R and Cp variable, as previously stated.

The velocity vector q (u, v, w, α, β) requires special consideration as to the axes of co-ordinates and the angular direc-

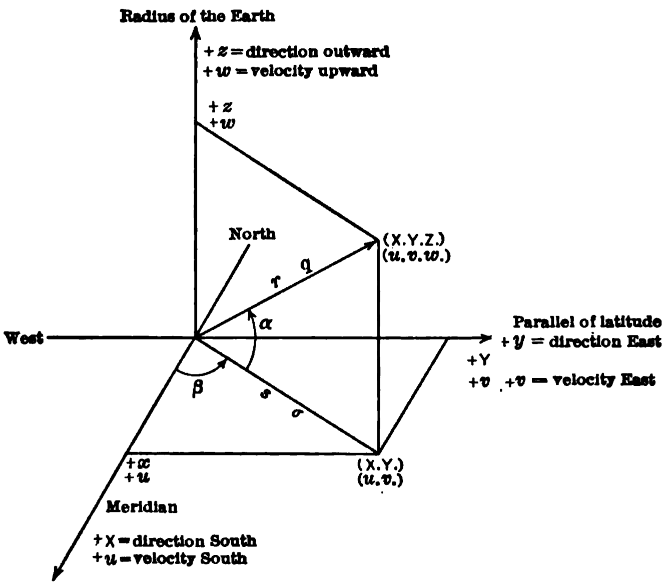

FIG. 2. The rectangular co-ordinate axes with component velocities and angles

tions. There is great confusion in meteorology in the manner of recording the motions of the atmosphere. The popular use of the compass points giving the directions from which the wind blows in the azimuth rotation N, E, S, W, is entirely inapplicable in meteorological computations. This system should be reversed in two respects, (1) The vector direction is that toward which the air moves, instead of that from which it blows, making an azimuth difference of 180°; (2) the proper co-ordinate axes make the azimuth rotation (S, E, N, W). The result of these two changes is effected by the formula.

(28) $$\beta = 360^\circ - A,$$

where A is the azimuth in degrees from the north through the east, and β *the azimuth from the south through the east*, the vector being changed to record the *direction "towards" instead of the direction "from"* which the wind moves. The rectangular co-ordinate axes are fixed by the common convention of a right-handed rotation about a radius of the earth with positive translation outward.

(Meridian) $+x$ = axis positive southward; $+u$ = velocity south.

(Parallel) $+y$ = axis positive eastward; $+v$ = velocity east.

(Radius) $+z$ = axis positive outward; $+w$ = vertical velocity.

(29) Horizontal Plane. $s = (x^2 + y^2)^{\frac{1}{2}}$. $\sigma = (u^2 + v^2)^{\frac{1}{2}}$.

(30) Vertical Plane. $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$. $q = (u^2 + v^2 + w^2)^{\frac{1}{2}}$.

(31) Azimuth angle (S, E, N, W), $\tan\beta = \dfrac{y}{x} = \dfrac{v}{u}$.

(32) Vertical angle (above horizon), $\tan\alpha = \dfrac{z}{s} = \dfrac{w}{\sigma}$.

The same co-ordinate relations should be employed in terrestrial magnetism, atmospheric electricity, and vector physics generally.

Besides recording the direction of motion very inconveniently, the velocities themselves usually require a series of transformations to reduce than to practical dynamics. The anemometers are commonly graduated in kilometers per hour, or in miles per hour, but they should be graduated in meters per second for the (M. K. S.) system, and in feet per second for the (F. P. S.) system in order to conform with the other terms of equation (27). In the electrical self-registration of the wind direction it is common to limit the compass points to eight in number, N, NE, E, SE, S, SW, W, NW, but in all problems requiring accurate wind deflecting components, as in studies of the diurnal convection, it is necessary to use at least 16 points of 22.5° each in order to compute the observed, resultant, and deflecting vectors. Finally, it would be much better to record the azimuth β (S, E, N, W) in degrees of arc, as can be readily done by a good mechanical device. The vertical angle α must be computed instead of observed, because it is small except in tornadoes and cannot be ordinarily measured mechanically. It is very important to record the wind vector (q, β, α) in the system thus described, in order to facilitate all studies in the higher problems. It may be noted that there is evidence to show that the wind velocity recorded by the anemometer is on a scale about 20 per cent. greater in the United States than in Europe. This subject should be fully tested as soon as possible. It is also known that the ordinary anemometer registers excessive velocities as compared with a force dynamometer, such that the recorded value 40 means 33, 60 means 48, 80 means 62, thus introducing great errors in the dynamic equations unless corrected.

The Constants and Formulas of Static Meteorology

Meteorology distributes itself into three parts in accordance with the requirements of equation (27). *Static* meteorology develops,

$$(33) \qquad - g\,(z_1 - z_0) = + \frac{P_1 - P_0}{\rho_{10}},$$

and it is that which is generally used in the construction of synoptic weather charts and the other elementary problems.

Dynamic meteorology develops,

$$(34) \qquad - g\,(z_1 - z_0) = + \frac{P_1 - P_0}{\rho_{10}} + \tfrac{1}{2}\,(q_1^2 - q_0^2),$$

and is concerned with the several general equations of motion connecting circulation and pressure.

Thermodynamic meteorology develops,

$$(27)\quad -g(z_1 - z_0) = +\frac{P_1 - P_0}{\rho_{10}} + \tfrac{1}{2}(q_1^2 - q_0^2) + (Q_1 - Q_0),$$

and unites the radiation with the circulation and the pressure through the functions of work and inner energy. It follows that the term $-g(z_1 - z_0)$ may be broken up into three parts:

$$(36)\quad -g(z - z_0) = -g(z_1 - z_0) - g(z_2 - z_1) - g(z_3 - z_2)$$

where we have, respectively,

$$(33)\quad -g(z_1 - z_0) = +\frac{P_1 - P_0}{\rho_{10}} \text{ the pressure effect,}$$

$$(37)\quad -g(z_2 - z_1) = +\tfrac{1}{2}(q_1^2 - q_0^2) \text{ the circulation effect,}$$

$$(38)\quad -g(z_3 - z_2) = +(Q_1 - Q_0) \text{ the radiation effect.}$$

Each of these terms is effective in disturbing the normal pressure, temperature, and density levels, which would assume fixed positions when uninfluenced by the absorption and the emission of solar and terrestrial radiation, the entire process being the means of continually returning to normal equilibrium.

In order to derive the constants and the formulas for static meteorology, the formulas (25) and (33) are united to form,

$$(39)\quad -\frac{P_1 - P_0}{\rho_{10}} = g_{10}(z_1 - z_0) = -Cp_{10}(T_1 - T_0),$$

where the mean gravity and the mean specific heat between the two vertical points, z_1 and z_0, are to be used. Then,

$$(40)\quad -(P_1 - P_0) = g_{10}\,\rho_{10}(z_1 - z_0) = -\rho_{10}\,Cp_{10}(T_1 - T_0).$$

Since for a column on a base of unit square area $\rho_{10}(z_1 - z_0) = M$, the mass that produces the pressure $-(P_1 - P_0)$ when acted upon by the force of gravity g_{10}, in the differential equation, is,

$$(41)\quad -\int dP = \int g\,dm = \int g\,\rho\,dz = -\int \rho\,Cp\,dT.$$

If the upper limit is at the top of the atmosphere, and the lower limit at the bottom of it, on the sea level in latitude 45°, and for the temperature $T = 273°$, (40) reduces to,

(42) $$P_0 = g_0 \rho_0 z_1 = \rho_0 Cp_0 (273° - T_1).$$

When the temperature of reduction is

(43) $$T_1 = 273°$$

the last form in (42) disappears. The constants of static meteorology conform to (42) for any substance whatsoever: water, mercury, dry air, aqueous vapor, or mixtures. If P_0 is the pressure of one standard atmosphere the density must change in an inverse proportion with the height. In the following notation,

Substance	Density	Height	Column
Water	ρ_w	h_w	Water column
Mercury	ρ_m	B_0	Barometer
Dry air	ρ_0	l_0	Homogeneous
Aqueous vapor	ρ_2	l_2	Vapor column

(42) becomes, specifically,

(44) $$P_0 = g_0 \rho_w h_w = g_0 \rho_m B_0 = g_0 \rho_0 l_0 = g_0 \rho_2 l_2.$$

and, (water) (mercury) (dry air) (aqueous vapor)

(45) $$\frac{P_0}{g_0} = \rho_w h_w = \rho_m B_0 = \rho_0 l_0 = \rho_2 l_2.$$

Before evaluating equations (44) and (45), it is necessary to adopt the standard constants* of transformation between the three fundamental systems (M. K. S.), (C. G. S.), (F. P. S.).

The equivalent units of length and volume are,

(46) 1 meter = 100 centimeters = 3.2809 feet.
1 meter³ = 1000000 cm³ = 35.3166 cu. ft.

The standard relation between volume and mass is,
1 cubic centimeter of water = 1 gram at the temperature 276.9°.

(47) 1 kilogram = 1000 grams = 2.20462 pounds.
1000 kilograms = 1000000 grams = 2204.62 pounds.

*The subject of units and physical constants can be studied in Everett's "Units and Physical Constants," Gray's "Smithsonian Physical Tables," and in the text-books on physics generally.

Hence, by division, the equivalents become,

$$(48)\quad 1000\,\frac{\text{kilograms}}{(\text{meter})^3} = 1\,\frac{\text{gram}}{(\text{centimeter})^3} = 62.4237\,\frac{\text{pounds}}{(\text{foot})^3},$$

$$16.0198\,\frac{\text{kilograms}}{\text{m}^3} = 0.0160198\,\frac{\text{gram}}{\text{cm}^3} = 1\,\frac{\text{pound}}{\text{ft.}^3}.$$

Three Series of Constants in Three Systems of Units

In Static Meteorology there are three series of constants for force units, mass units, and heat units in the three systems of units (M. K. S. C°), (C. G. S. C°), (F. P. S. F°). These develop from fundamental principles or definitions. Thus, to illustrate by pressure:

1. Force pressure = mass × acceleration.

$$(49)\qquad P_0 = \rho_m B_0 g_0 = \rho_0 l_0 g_0 = M g_0 = p_0 g_0 = p_A \frac{g_0}{A}.$$

2. Mass pressure = heat pressure × mechanical equivalent of heat.

$$(50)\qquad p_0 = M = \frac{P_0}{g_0} = \frac{p_A}{A}.$$

3. Heat pressure = force pressure × heat equivalent of gravity work.

$$(51)\qquad p_A = A\,M = P_0 \frac{A}{g_0} = A\,p_0.$$

These transformations apply to the heat terms R, Cp, Cv, in the several systems. These factors become in the several unit systems:

TABLE 2

Gravity and Mechanical Equivalents of Heat

Work and Heat Equivalents	(M. K. S.)	(C. G. S.)	(F. P. S.)
g_0 Acceleration of gravity.....	9.8060	980.60	32.173
$\frac{1}{A}$ Work equivalent of heat....	426.887	42683.7	777.93
$\frac{g_0}{A}$ Gravity-work of heat......	4185.1	41851000	25028.2
A Heat equivalent of work...	0.002343	0.00002343	0.0012855
$\frac{A}{g_0}$ Heat equivalent of gravity work..................	0.00023894	0.000000023894	0.000039954

A × (Heat in mechanical units) = Heat units of heat = calories.

In Tables 3, 4, and 5 have been collected together the constants in the three unit systems. They illustrate practically the formulas (1), (17), (18), (22), and (44) in Table 3, (50) in Table 4, and (51) in Table 5. By combining these constants and formulas a very large amount of static meteorology is derived.

TABLE 3

Three Series of Constants in Three Systems of Units

(1) Gravitational Force Units

Formulas	S	(M. K. S. C°) Meter-Kg-Second		(C. G. S. C°) Cm.-Gram-Second°		(F. P. S. F°) Foot-Pound-Second	
			Log.		Log.		Log.
Gravity	g_0	9.8060	0.99149	980.60	2.99149	32.178	1.50749
Density of water	ρ_w	1000.0	3.00000	1.0000	0.00000	62.4237	1.79535
Height	h_w	10.3329	1.01422	10332.9	3.01422	33.901	1.53021
(44) $P_0 = g_0 \rho_w h_w$	P_0	101323.5	5.00571	1013235.	6.00571	68085	4.83305
Gravity	g_0	9.8060	0.99149	980.60	2.99149	32.178	1.50749
Density of mercury	ρ_m	13595.8	4.13340	13.5958	1.13340	848.70	2.92875
Barometer height	B_0	0.760	9.88081	76.0	1.88081	2.4935	0.39681
(44) $P_0 = g_0 \rho_m B_0$	P_0	101323.5	5.00571	1013235.	6.00571	68085	4.83305
Gravity	g_0	9.8060	0.99149	980.60	2.99149	32.178	1.50749
Density of air	ρ_0	1.29305	0.11162	.00129305	7.11162	0.080717	8.90696
Height	l_0	7991.04	3.90260	799104.	5.90260	26218.1	4.41860
(44) $P_0 = g_0 \rho_0 l_0$	P_0	101323.5	5.00571	1013235.	6.00571	68085.	4.83305
Gravity	g_0	9.8060	0.99149	980.60	2.99149	32.178	1.50749
Density aqueous vapor	ρ_2	0.80427	9.90540	.00080427	6.90540	0.056009	8.69974
Height	l_2	12847.6	4.10882	1284760.	6.10882	42249.1	4.62582
(44) $P_0 = g_0 \rho_2 l_2$	P_0	101323.5	5.00571	1013235.	6.00571	68085.	4.83305
Specific heats $\frac{C_p}{C_v}$	k	1.4062486	0.14806	1.4062486	0.14806	1.4062486	0.14806
Ratio $\frac{C_p}{C_v} - 1$	$k - 1$	0.4062486	9.60879	0.4062486	9.60879	0.4062486	9.60879
(17) Const. press.	$\frac{k}{k-1}$	3.461545	0.53927	3.461545	0.53927	3.461545	0.53927
(18) Const. vol	$\frac{1}{k-1}$	2.461545	0.39121	2.461545	0.39121	2.461545	0.39121
The Boyle-Gay Lussac	R_0	287.0334	2.45793	2870334.	6.45793	1716.52	3.23465
Law	T_0	273.	2.43616	273.	2.43616	491.4	2.69144
	ρ_0	1.29305	0.11162	0.00129305	7.11162	0.080717	8.90696
(1) $P_0 = \rho_0 R_0 T_0$	P_0	101323.5	5.00571	1013235.	6.00571	68085	4.83305
$l_0 g_0 = R_0 T_0$	l_0	7991.04	3.90260	799104.	5.90260	26218.1	4.41860
(1) and (44)	g_0	9.8060	0.99149	980.60	2.99149	32.178	1.50749
$a = \frac{1}{T_0}$	a	0.003663	7.56384	0.003663	7.56384	0.002035	7.30856
$R_0 = l_0 g_0 a$	R_0	287.0334	2.45793	2870334	6.45793	1716.52	3.23465
(17) $C_p = R_0 \frac{k}{k-1}$	C_{p_a}	993.5787	2.99720	9935787.	6.99720	5941.86	3.77392
(18) $C_v = R_0 \frac{1}{k-1}$	C_{v_a}	706.5453	2.84914	7065453.	6.84914	4225.34	3.62586
(22) $-\frac{dT}{dz} = \frac{g_0}{C_{p_a}}$	a_0	0.0098695	7.99429	.000098695	5.99429	0.0054146	7.73357

TABLE 4

(2) Mass or Units of Weight

Formulas	S	(M. K. S. C°) Meter-Kilgm.-Sec.	Log.	(C. G. S. C°) Cm.-Gm.-Sec.	Log.	(F. P. S. F°) Ft.-Pound-Sec.	Log.
(50) $p = P_0/g_0$........	p	10332.8	4.01422	1033.28	8.01422	2116.20	3.82556
$R = p/\rho_0 T_0$......	R	29.2713	1.46644	2927.13	8.46644	53.353	1.72716
$Cp = R \frac{k}{k-1}$......	Cp	101.3235	2.00571	10132.35	4.00571	184.683	2.26643
$Cv = Cp - R$.....	Cv	72.0522	1.85765	7205.22	8.85765	131.330	2.11837

TABLE 5

(3) Heat Units

Formulas	S	(M. K. S. C°)		(C. G. S. C°)		(F. P. S. F°)	
Work equiv. heat......	$\frac{1}{A}$	426.837	2.63022	42683.7	4.63022	777.93	2.89094
Heat equiv. work......	A	0.002343	7.36978	0.00002343	5.36978	0.0012855	7.10906
(51) $p_A = A\,p$.......	p_A	24.2106	1.38400	0.024106	8.38400	2.72025	0.43461
$R_A = A\,R$......	R_A	0.068583	8.83622	0.068583	8.83622	0.068583	8.83622
$Cp_A = A\,Cp$.....	Cp_A	0.237406	9.37549	0.237406	9.37549	0.237406	9.37549
$Cv_A = A\,Cv$....	Cv_A	0.168823	9.22743	0.168823	9.22743	0.168823	9.22743

Work and Heat Units

One large calorie is the heat required to raise 1 kilogram of water from 0° to 1° C.

One small calorie (therm.) is the heat required to raise 1 gram of water from 0° to 1° C.

One British thermal unit is the heat required to raise 1 pound of water from 32° to 33° F.

One calorie = 1000 therms = 3.968 Br. th. u. = 426.837 kilogram meters.

One therm = 0.003968 Br. th. u. (3.968 = 2.2046 × 1.8.)

One dyne is the force which acting upon a gram for one second generates a velocity of one centimeter per second; it produces the C. G. S. unit of acceleration on one gram; it produces the C. G. S. unit of momentum on any mass per second.

One erg is the amount of work done by one dyne working through the distance of one centimeter; it is the C. G. S. unit of energy.

One erg $= 1$ centimeter dyne $= \frac{1}{980.60} = 0.0010198$ gram cm.

$= \frac{1}{980.60 \times 1000 \times 100} = 0.000000010198$ kilogram meter.

One large calorie $= 1$ kilogram-degree C° water $= 426.837$ kilogram meters

$= 426.837 \times 980.60 \times 1000 \times 100 = 4.1851 \times 10^{10}$ ergs, C. G. S.

One small calorie $= 1$ gram-degree C° water $= 426.837$ gram meters

$= 426.837 \times 980.60 \times 100 = 4.1851 \times 10^{7}$ ergs, C. G. S.

One British thermal unit $= 1$ pound-degree F° water

$= 426.837 \times \frac{3.2809}{1.8} = 777.93$ foot-pounds.

Work to Heat

$\frac{1}{A}$. The mechanical equivalent of heat is the work required by work-friction to produce the given heat.

		Log.		
(52)	$\frac{1}{A} = 426.837$	2.63022	kilogram meters	(M. K. S.)
	$= 42683.7$	4.63022	gram centimeters	(C. G. S.)
	$= 777.93$	2.89094	foot-pounds	(F. P. S.)
(53)	$\frac{g_0}{A} = 4185.1$	3.62171	joules	(M. K. S.)
	$= 41851000.$	7.62171	ergs	(C. G. S.)
	$= 25028.2$	4.39843	absolute units	(F. P. S.)

Heat to Work

A. The heat equivalent of work is the heat that is required to do a given amount of work.

		Log.		
(54) A	$= 0.002343$	7.36978	kilogram calorie	(M. K. S.)
	$= 0.00002343$	5.36978	gram therm	(C. G. S.)
	$= 0.0012855$	7.10906	Br. th. units	(F. P. S.)
(55) $\frac{A}{g_0}$	$= 0.00023894$	6.37829		(M. K. S.)
	$= 0.000000023894$	2.37829		(C. G. S.)
	$= 0.000039954$	5.60157		(F. P. S.)

I. For the Same Point or Station

Variations from the Standard P_0, ρ_0, R_0, T_0

The several formulas derived from $P_0 = \rho_0 R_0 T_0$ apply only to the sea level on latitude 45°, but the variations, P_1, ρ_1, R_1, T_1, are incessant in the earth's atmosphere, and the formulas must be derived for passing from one condition to another. Meteorology divides itself into two main branches according as R is taken constant or variable, and it is a principal part of this work to discuss the formulas when R is variable from one stratum to another. When the point of observation is on the sea level or on the lower land areas, it is proper to assume $R_1 = R_0$ constant, which greatly simplifies the computations. If the variations occur at the same place or station, g_0 is also constant. For two variable conditions of dry air at a given place we may write the two in a ratio in several forms, using (44), (49), (50), (51):

$$(56)\quad \frac{P}{P_0} = \frac{p\, g_0}{p_0 g_0} = \frac{p_A \frac{g_0}{A}}{p_{A0} \frac{g_0}{A}} = \frac{M\; g_0}{M_0\, g_0}. \quad \text{Hence,}$$

$$(57)\quad \frac{P}{P_0} = \frac{p}{p_0} = \frac{p_A}{p_{A0}} = \frac{\rho\; T\; R_0}{\rho_0\, T_0\, R_0} = \frac{\rho_m\, B\; g_0}{\rho_m\, B_0\, g_0} = \frac{\rho\; l\; g_0}{\rho_0\, l_0\, g_0}.$$

We shall confine our attention chiefly to the force pressures P, P_0 in the (M. K. S.) system, and to the first series of constants in Table 3. There are numerous equivalents which are easily derived from the formulas.

(58) The pressure ratio, $\frac{P}{P_0} = \frac{\rho T}{\rho_0 T_0} = \frac{\rho l}{\rho_0 l_0} = \frac{B}{B_0}.$

The temperature ratio becomes,

(59) $\frac{T}{T_0} = \frac{273 + t}{273} = 1 + \frac{1}{273} t = (1 + \alpha t).$

The pressure ratio may have several forms, from which are derived the pressure-density ratio,

(60) $\frac{P}{\rho} = \frac{P_0}{\rho_0} \frac{T}{T_0} = \frac{P_0}{\rho_0} (1 + \alpha t).$ *Auxiliaries.*

$= \frac{B_0 \rho_m g_0}{\rho_0} (1 + \alpha t).$ $P_0 = B_0 \rho_m g_0.$

$= P_0 v_0 (1 + \alpha t).$ $\frac{1}{\rho_0} = v_0.$

$= R_0 T_0 (1 + \alpha t).$ $P_0 v_0 = R_0 T_0.$

(61) The temperature ratio, $\frac{T}{T_0} = \frac{P}{P_0} \cdot \frac{\rho_0}{\rho}.$

(62) The density ratio, $\frac{\rho}{\rho_0} = \frac{P}{P_0} \cdot \frac{T_0}{T}.$

The temperature varies as the pressure and inversely as the density; the density varies as the pressure and inversely as the temperature.

II. For Different Points on Any Vertical Line z

The Variations of Gravity, Density, Temperature, and Pressure

The practical problems in static meteorology consist to a considerable extent of the reduction of barometric pressures from one elevation to another along a radius of the earth extended, or inversely the determination of the difference of elevation between two measured barometric pressures along the same vertical. The former process is applied in forming the synchronous charts of pressure reduced to the sea level, or to any other adopted plane which are used in public forecast charts of storm and weather conditions, and the latter to preliminary surveys in mountain and plateau regions. The entire process is, however, very complex in its application to particular cases, and it requires much experience in managing the details of the computations. It will be necessary to describe somewhat fully the several terms that are to be integrated from one level to another, which enter the final barometric and hypsometric formulas. Those here developed are such as are found in the Standard Treatises, but another method of reduction will be introduced at a later section. The first problem to describe is the gravity value in any latitude, ϕ, and at any height above the sea level, z; the second considers the density of the air as a mixture of gases and aqueous vapor in varying proportions; the third is to determine the temperature gradients in a vertical direction in the free air and within the land masses; and the fourth is the use of the barometer as an instrument of precision, together with the discussion of the observed heights of the mercury column. All the details easily found in good works on meteorology will be very briefly mentioned.

I. The Acceleration of Gravitation $\int g_{\phi z}\, dz$.

From formula (4), which is of geodetic origin,

(63) $g_\phi = g_0\,(1 - 0.00260 \cos 2\phi)$, the latitude variation.

(64) $g_z = g_\phi \left(1 - \frac{2z}{R}\right)$, the elevation variation. Hence,

(65) $$\int g_{\phi z}\, dz = 9.8060\,(1 - 0.00260 \cos 2\phi) \int_{z_0}^{z} \left(1 - \frac{2z}{R}\right) dz.$$

The force of gravity varies inversely as the square of the distance from the center of the earth.

(66) $$\frac{g_z}{g_0} = \frac{R^2}{(R+z)^2} = \frac{R^2}{R^2 + 2zR + z^2} = 1 - \frac{2z}{R}.$$

(67) $$\int_{z_0}^{z} g_z\, dz = \int_{z_0}^{z} \left(1 - \frac{2z}{R}\right) dz = (z - z_0) - \frac{z^2 - z_0^2}{R} = \left(1 - \frac{z + z_0}{R}\right)(z - z_0).$$

The radius of the earth may be taken in the mean from Bessel's spheroid,

(68) R_m = 6370191 meters, 6.8041525 log.
R_f = 20899600 feet, 7.3201380 log.

The computed values of formula (4), without any integration in latitude and altitude, are given for a few selected points in Table 6.

TABLE 6

EVALUATION OF FORMULA (4)

$$g_{\phi z} = 9.8060\ (1 - 0.00260 \cos 2\,\phi)\left(1 - \frac{2z}{R}\right)$$

z	90°	80°	70°	60°	50°	45°	40°	30°	20°	10°	0°
Meters.	m.	m.	m.	m.	m.	m.	m.	m.	m.	m.	m.
20000	9.7703	9.7688	9.7643	9.7574	9.7489	9.7444	9.7399	9.7314	9.7245	9.7200	9.7185
15000	9.7857	9.7842	9.7797	9.7728	9.7643	9.7598	9.7553	9.7468	9.7399	9.7354	9.7339
10000	9.8011	9.7996	9.7951	9.7882	9.7797	9.7752	9.7707	9.7622	9.7553	9.7508	9.7493
5000	9.8165	9.8150	9.8105	9.8036	9.7951	9.7906	9.7861	9.7776	9.7707	9.7662	9.7647
000	9.8319	9.8304	9.8259	9.8190	9.8105	9.8060	9.8015	9.7930	9.7861	9.7816	9.7801

The variation in height $g_0 \dfrac{2z}{R}$ = 0.00308 meters per 1,000 meters, and 0.00308 feet per 1000 feet for g_0 = 32.173 feet.

There has been a discussion as to the effect of the land masses upon the action of gravity, whether the coefficient in the formula should be 2.00, as developed in (66), or be modified. Ferrel claims that it should remain 2.00; the Smithsonian Meteorological Tables have adopted 1.96; and the International Meteorological Tables have taken 1.25, which latter value is here adopted. The plateau regions of North and South America, Asia, and Africa will be best represented by $1.25\,\dfrac{z}{R}$, where z has a considerable value, reckoned rom the sea level. In balloon ascensions from the ocean or from the low plains it may be better to increase the value to 2.00, but this can be determined from observations by means of mercurial and aneroid barometers. In fact, the aneroid barometer, when perfectly adjusted as a mechanism, measures the local hydrostatic pressure without any gravity factor. Among the computations to be introduced in a later section there will be examples of this action. Admitting the coefficient 1.25, we have

$$(69)\quad g_z = g_0\left(1 - 1.25\frac{z}{R}\right) = 9.806\ (1 - 0.000000196\,z) \text{ metric.}$$
$$= 32.173\ (1 - .0000000598\ z) \text{ English.}$$
$$= 9.806 - 0.00000192\ z \text{ metric.}$$
$$= 32.173 - 0.00000192\ z \text{ English.}$$

Similarly, the correction for height can be applied to any other value of g_ϕ , as found on the lower line of Table 6.

2. *The Density of the Atmosphere as a Mixture of Several Constituent Gases*

For the practice of barometry it is sufficient to take account of the atmosphere as a mixture of dry air, aqueous vapor and carbon dioxide, commonly called carbonic acid.* In physical problems there are the gases oxygen, nitrogen, hydrogen, carbonic oxide, and traces of argon, helium, neon, krypton. We shall summarize the treatment of several gases in a mixture and the data of the kinetic theory of gases in this connection.

Adopt the notation for standard conditions as expressed by $P_0 = \rho_0 R_0 T_0$, or $P_0 v_0 = R_0 T_0$.

	Mixture.	*Dry Air.*	*Aqueous Vapor.*	*Carbonic Acid.*
Density	ρ_{m0}	ρ_{10}	ρ_{20}	ρ_{30}
Volume	v_{m0}	v_{10}	v_{20}	v_{30}

The general equation for mixture is,

$$(70)\quad \rho_{m0}\, v_{m0} = \rho_{10}\, v_{10} + \rho_{20}\, v_{20} + \rho_{30}\, v_{30}.$$

The values to be assigned to the terms are:

Dry air $\quad \rho_{10} = 1.29278. \quad v_{10} = v_{m0} - v_{20} - v_{30}.$

* In view of the possible existence of the real carbonic acid (H_2CO_3) at the low temperature of the isothermal layer, the use of the word "acid" instead of anhydride can not be commended.

Aqueous vapor $\rho_{20} = 0.622\ \rho_{10}$. v_{20} = variable amounts.
Carbonic acid $\rho_{30} = 1.529\ \rho_{10}$. $v_{30} = 0.0004\ v_{m0}$.
Mixture ρ_{m0} to be computed. $v_{m0} = v_{10} + v_{20} + v_{30}$.
Introducing these values in (70) and dividing by v_{m0}

$$(71)\quad \rho_{m0} = 1.29278\left(1 - \frac{v_{20}}{v_{m0}} - \frac{v_{30}}{v_{m0}} + 0.622\frac{v_{20}}{v_{m0}} + 1.529\frac{v_{30}}{v_{m0}}\right),$$

$$= 1.29278\left(1 - 0.378\frac{v_{20}}{v_{m0}} + 0.529\frac{v_{30}}{v_{m0}}\right).$$

Since the ratio of the volume of the constituent to the volume of the mixture is the same as that of the partial pressure of the constituent to the total pressure of the mixture, we have generally

$$(72)\quad \frac{v_{n0}}{v_{m0}} = \frac{p_{n0}}{p_{m0}} = \frac{p_n}{p_m} = \frac{e}{B},$$

where p_{n0} and p_{m0} are for the normal data $(P_0\ T_0)$,
p_n and p_m are for any $(P\ T)$,
e and B are the barometric pressures.

Hence, $\frac{v_{20}}{v_{m0}} = \frac{e_0}{B_0}$, and $\frac{v_{30}}{v_{m0}} \doteq 0.0004$, so that,

$$(73)\quad \rho_{m0} = 1.29278\left(1 - 0.378\frac{e_0}{B_0} + 0.00021\right).$$

It is customary to unite the terms for the dry air and the carbonic acid in the normal density,

$$(74)\quad \rho_0 = 1.29278 + 0.00027 = 1.29305 \text{ per cubic meter.}$$

In (73) e_0 is to be taken in meters of mercury in the (M. K. S.) system, in millimeters in the (C. G. S.) system, and in feet in the (F. P. S.) system. Since e_0 varies incessantly in the atmosphere no fixed value can be assigned to it on any level. The reduction from the normal $(P_0\ \rho_{m0}\ T_0)$ to any other condition $(P\ \rho_m\ T)$ on the same level z_0 is given by (60), substituting ρ_m for ρ, and ρ_{m0} for ρ_0,

$$(75)\quad \rho_m = \frac{P}{P_0}\frac{T_0}{T}\rho_{m0} = \frac{P}{P_0}\frac{T_0}{T}\rho_{m0}\left(1 - 0.378\frac{e}{B_0}\right),$$

$$(76)\quad \rho_m = \frac{B}{B_0}\,\rho_{m0}\,\frac{1}{(1 + 0.00366\,t)}\,\frac{1}{\left(1 + 0.378\frac{e}{B_0}\right)}.$$

Since we retain the density $\rho_0 = 1.29305$ for the value of ρ_{m0}, with dry air at normal pressure in this equation, the corresponding barometric pressure in the fraction expressing the partial pressure of aqueous vapor must be B_0; but e may be any pressure whatever, according to the dryness of the air.

The Integral Mean $\int_{z_0}^{z} 0.378\,\frac{e}{B_0}$.

The pressure of the aqueous vapor decreases from the ground upward in a geometric ratio, which is expressed approximately on the average by the formula,

$$(77)\quad e = e_0\,10^{-\frac{z}{6517}} \text{ in meters, } e = e_0^{-\frac{z}{21381}} \text{ in feet.}$$

It will be shown that the barometric pressure diminishes by a similar law,

$$(78)\quad B = B_0\,10^{-\frac{z}{18400}} \text{ in meters, } B = B_0\,10^{-\frac{z}{60367}} \text{ in feet.}$$

By combining these in the ratio $\frac{e}{B}$ it becomes,

$$(79)\quad \frac{e}{B} = \frac{e_0}{B_0}\,10^{-\frac{z}{10091}} \text{ in meters, } \frac{e}{B} = \frac{e_0}{B_0}\,10^{-\frac{z}{33108}} \text{ in feet.}$$

These can be reduced from the common base 10 to the Naperian base ε by the modulus $M = 0.43429$.

$$10091 \times M = 4383,\quad 33108 \times M = 14378.$$

The expression for the integral mean from z_0 to z is

$$(80)\quad \beta = \frac{1}{z - z_0}\,0.378\,\frac{e_0}{B_0}\int_{z_0}^{z} \varepsilon^{-\frac{z}{4383}}\,dz \text{ for the metric system.}$$

$$(81)\quad \beta = \frac{1}{z - z_0}\,0.378\,\frac{e_0}{B_0}\int_{z_0}^{z} \varepsilon^{-\frac{z}{14378}}\,dz \text{ for the English system.}$$

These can be developed in a series, as shown in the "Report on the International Cloud Observations," U. S. W. B., 1898, page 491,

$$(82)\quad \beta = 0.378\frac{e_0}{B_0}\left[1 - \frac{1}{2!}\left(\frac{z}{4383}\right) + \frac{1}{3!}\left(\frac{z}{4383}\right)^2 - \frac{1}{4!}\left(\frac{z}{4383}\right)^3 + \ldots\right] \text{ metric.}$$

$$(83)\quad \beta = 0.378\frac{e_0}{B_0}\left[1 - \frac{1}{2!}\left(\frac{z}{14378}\right) + \frac{1}{3!}\left(\frac{z}{14378}\right)^2 - \frac{1}{4!}\left(\frac{z}{14378}\right)^3 + \ldots\right] \text{ English.}$$

That is to say, having the vapor pressure e_0 and the barometric pressure B_0 at the surface one can compute the average value of the integral of the term $0.378\frac{e}{B}$ up to the height z. It is commonly impractical to measure the values of e and B at several points in the atmosphere, and for many computations this method of mean integration upward from the surface is quite sufficient for practical work. Also, it is a very expeditious method when using the humidity table 92, page 548, for metric measures and Table 19 of the "Barometry Report," U. S. W. B., 1900, page 108, for English measures. The mean values of $\frac{e}{B}$ for the air column is often taken as the arithmetical mean of the observed values at an upper station z and a lower station z_0. In the case of balloon and kite ascension, the registered relative humidity, temperature, and pressure can be computed to the integral mean value required. When only temperature and pressure are registered, this correction to the density of the atmosphere is not available in the hypsometric formula.

The General Formulas for the Mixture of Gases

The general principles controlling the mixture of gases are so often useful in meteorology that it will be convenient to collect together the common formulas expressing the several processes. It will now be proper to pass from the system of M. K. S. units to the system of C. G. S. units, and to bring forward the terms applicable to the thermodynamics of the atmosphere and the kinetic theory of gases. We adopt the following notation:

a = atomic weight.

m = molecular weight $= \frac{K}{R}$.

(84) K = the absolute gas constant $= m\,R = \frac{P\,m}{T\,\rho}$.

Compute K in the (C. G. S.) system, using the gas hydrogen, $m_H = 2$, $\rho_H = 0.000089996$. Log = 5.95422 − 10.

$$K = \frac{1013235 \times 2}{273 \times 0.000089996} = 82482000.\ \frac{dyne}{cm^2}.\quad \text{Log} = 7.91636.$$

$$K\frac{A}{g_0} = \frac{82482000}{41851000} = 1.9708 \text{ small calories or therms.}$$

Using the values for air $m_0 = 28.736$, $\rho_0 = 0.00129305$, the same result is obtained. This formula applies to the three systems and to all gases. The density and molecular weight of hydrogen are related to those of other gases so that,

(85) $$m\,\rho_H = m_H\,\rho,$$

and for this reason hydrogen is the standard.

n = the number of molecules in a unit volume.

N = the number of molecules in a V-volume.

$M = N\,m$ = Mass.

(86) Number. $$N = \frac{M}{m} = V\,n = \frac{P\,V}{K\,T}.$$

(87) Mass. $$M = N\,m = V\,n\,m = \frac{P\,V\,m}{K\,T} = \frac{P\,V}{R\,T} = \frac{P}{R\,T}\cdot\frac{M}{n\,m} = \frac{P\,M}{R\,T\,\rho}.$$

(88) Volume. $$V = M\,v = \frac{M}{\rho} = \frac{M}{n\,m} = \frac{M}{m}\frac{K\,T}{P} = N\frac{K\,T}{P} = M\frac{R\,T}{P}.$$

(89) Pressure. $P = \frac{N K T}{V} = n K T = \frac{M}{m}\frac{K T}{V} = \frac{M}{V} R T =$

$n m R T = \rho R T.$

(90) Density. $\rho = \frac{1}{v} = n m = \frac{M}{V} = \frac{m}{K}\frac{P}{T} = \frac{P}{R T}.$

(91) Volume of unit mass. $v = \frac{1}{\rho} = \frac{1}{n m} = \frac{V}{M} = \frac{K}{m}\frac{T}{P} = \frac{R T}{P}.$

(92) Constant. $R = \frac{K}{m} = \frac{P V}{M T} = \frac{P}{T n m} = \frac{P}{T \rho}.$

Referring to the standard gas hydrogen there are some special values for the gas constant.

Logs.

(93) $R_H = R\frac{\rho}{\rho_H} = R\frac{v_H}{v} = R\frac{m}{m_0} = 41241000 \cdot \frac{\text{gram cm.}^2}{\text{sec.}^2\text{ temp.}}$ 7.61533

(94) $(R) = R_H \rho_H = R \rho = \frac{K \rho_H}{m_H} = 3711.5 \frac{\text{gram}^2}{\text{cm. sec.}^2\text{ temp.}}$ 3.56955

(95) $K \doteq \frac{(R) m_H}{\rho_H} = \frac{P_0 m_H}{\rho_H T_0} = 82482000 \text{ gram} \times$

$\frac{\text{cm.}^2}{\text{sec.}^2} \frac{\text{mol. wt.}}{\text{temp.}}$ 7.91636

If R_T is the gas constant for the heat energy at T,
R_m is the gas constant for the molecular energy,
R_a is the gas constant for the atomic energy. Then,

(96) $R_1 = R_T + R_m + R_a = \frac{3}{2}R = \frac{\overline{T}}{T} = \frac{\text{the mean kinetic energy}}{\text{absolute temperature}}.$

(97) $R_2 = \frac{5}{3} R_1 - R = Cv$ the specific heat at constant volume.

(98) $R_3 = \frac{5}{3} R_1 = Cp$ the specific heat at constant pressure.

These are related to the potential and kinetic energies in the following relations, and thence to the specific heats at constant pressure and constant volume.

Inner Potential Energies

(99) $J = J_m + J_a$.* The inner potential energy of molecules and atoms.

(100) $J_m = \frac{3}{2} P v = \frac{3}{2} \frac{P}{\rho} = \frac{3}{2} R T$. Inner molecular potential energy.

$J_a =$ Inner atomic potential energy.

$J_0 = \frac{2}{3} \overline{V}_i$ (inner viriol). Initial inner potential energy.

(101) $\overline{V} = -\frac{1}{2} \Sigma (X_x + Y_y + Z_z)$ the mean viriol for the force (X. Y. Z.).

Inner Kinetic Energies

(102) $H = H_m + H_q$. The inner kinetic energy of molecules and atoms.

$H_m =$ Inner molecular kinetic energy.

$H_a =$ Inner atomic kinetic energy.

Total Heat and Work Energies

$Q =$ The heat or total inner energy $= Cv\, T$.

$W =$ the work or total external potential energy.

(103) $Q = J + H + W = \left(\frac{5}{3} R_1 - R\right) T + J_0$. Total inner energy.

(104) $W = Pv = \frac{P}{\rho} = R T = \frac{2}{3} \overline{V}_e$ (outer viriol). External potential energy.

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*J_m and J_a relate to the trifling rearrangements of parts which are the only changes that can occur in ordinary chemical and physical reactions. We can not attack the enormous stores of energy shut up within the atoms.

(105) $\bar{U} = J - J_0 + W = J_m + J_a - J_0 + W.$ Potential energy.

(106) $\bar{U} - W = J - J_0.$ Accession of inner potential energy.

(107) $\bar{V} = H = -\frac{1}{2}\Sigma(X_x + Y_y + Z_z) = \frac{3}{2}\bar{U}$ = mean viriol or work done.

The *gas constants* are again defined thus:

(108) $R_T = \frac{H}{T}$ (heat) = the ratio of the inner kinetic energy to T.

(109) $R_m = \frac{1}{T}\frac{1}{2} m\bar{q}^2$ where $\bar{q}^2$ = the mean square velocity.

(110) $R_a = \frac{1}{T}\frac{1}{2}n\ m_1 q_1^2$, n = the number of atoms in a molecule.

The Specific Heats of Monatomic Gases

(111) $Cp = \frac{5}{3}R_1 - R + R = R_2 + R = \frac{5}{3}R_1 = \frac{5}{2}R.$

$$R = \frac{2}{3}R_1 = \frac{2}{3}R_2$$

(112) $Cv = \frac{5}{3}R_1 - R = \frac{5}{3}R_1 - \frac{2}{3}R_1 = R_1 = \frac{3}{2}R = R_2.$

$$\frac{3}{2}R = R_1 = R_2$$

(113) $\frac{Cp}{Cv} = \frac{R_2 + R}{R_2} = \frac{5}{3} = 1.67 = k$ for monatomic gases (mercury).

(114) $\frac{Cp}{Cp - Cv} = \frac{R_2 + R}{R} = \frac{k}{k-1} = \frac{5}{2}.$ Compare (17), (19).

(115) $\frac{Cv}{Cp - Cv} = \frac{R_2}{R} = \frac{1}{k-1} = \frac{3}{2}.$ Compare (18), (20).

(116) $Cp - Cv = R = \frac{2}{3}R_1.$ Compare (15), (16).

(117) $\frac{J_m}{Q} = \frac{\frac{3}{2}RT}{Cv\,T} = \frac{3}{2}\frac{R}{Cv} = \frac{3}{2}\frac{Cp - Cv}{Cv}.$

(118) $\frac{J_a}{Q} = 1 - \frac{J_m}{Q} = 1 - \frac{3}{2}\,\frac{Cp - Cv}{Cv} = \frac{5\,Cv - 3\,Cp}{2\,Cv}.$

The above formulas are in mechanical units, and they can be transposed into heat units on multiplying them by A = the heat equivalent of work which is $A = 0.00002343$ in the C. G. S. system by Table 2.

The Fundamental Laws of Physics

(119) Boyle's (Mariotte) Law. $P_0\,v_0 = P\,v = \left(\frac{P}{\rho}\right)_T = R_0\,T =$ constant for constant T.

(120) Gay Lussac's Law. $\left(\frac{P}{T}\right)_v = \frac{R}{v} =$ constant for constant v.

(121) Boyle-Gay Lussac Law. $\frac{P\,v}{T} = R =$ constant in perfect gases.

(122) Avogadro's Law. $N = \frac{M}{m} = \frac{P\,V}{K\,T} =$ constant for (P. V. T.) constant.

(123) Clausius' Law. $\frac{1}{2}\,m_1\,q_1^2 = \frac{1}{2}\,m_2\,q_2^2 =$ constant kinetic energy.

(124) Dalton's Law. $P = P_1 + P_2 + \ldots = \frac{2}{3}\,(\frac{1}{2}\,m_1\,q_1^2 + \frac{1}{2}\,m_2\,q_2^2 + \ldots)$. Pressures.

In ordinary gases as distinguished from perfect and ideal gases, all these formulas are more or less defective on account of the internal action of the atoms and the molecules upon each other under the stresses of electrical and other mechanical forces. Many formulas, with constant coefficients and exponents, have been devised to take account of these physical variations, but they will not be further mentioned in this place.

The Formulas for the Mixture of Several Gases

(125) Pressure. $P = \rho\,R\,T = \frac{R\,T}{v} = n\,m\,R\,T = n\,K\,T = \rho\,\frac{K}{m}\,T = \frac{R\,T\,M}{V}.$

(126)
$$P = P_1 + P_2 + P_3 + \ldots = (n_1 + n_2 + n_3 + \ldots) K T = \frac{V_1 + V_2 + \ldots}{V} P.$$

(127)
$$P_1 = \frac{n_1}{n_1 + n_2 + \ldots} P.$$
$$P_2 = \frac{n_2}{n} P. \qquad P_3 = \frac{n_3}{n} P.$$

(128)
$$P_1 : P_2 : P_3 \ldots = V_1 : V_2 : V_3 \ldots = R_1 M_1 : R_2 M_2 : R_3 M_3 \ldots *$$

(129)
$$P V = (M_1 R_1 + M_2 R_2 + \ldots) T = \left(\frac{M_1}{m_1} + \frac{M_2}{m_2} + \ldots\right) K T = (n_1 + n_2 + \ldots) K T.$$

(130) Mass.
$$\frac{M}{m} = \frac{M_1}{m_1} + \frac{M_2}{m_2} + \frac{M_3}{m_3} + \ldots$$

(131)
$$m = \frac{M_1 + M_2 + M_3 + \ldots}{n} = \frac{M_1 + M_2 + M_3 + \ldots}{\frac{M_1}{m_1} + \frac{M_2}{m_2} + \frac{M_3}{m_3} + \ldots}$$

(132)
$$n_1 : n_2 : n_3 \ldots = \frac{M_1}{m_1} : \frac{M_2}{m_2} : \frac{M_3}{m_3} \ldots = P_1 : P_2 : P_3 \ldots$$

(133) Density.
$$n m = n_1 m_1 + n_2 m_2 + n_3 m_3 + \ldots = \rho_1 + \rho_2 + \rho_3 + \ldots = \rho = \frac{1}{v}.$$

(134)
$$v = \frac{1}{\rho} = \frac{1}{\frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \ldots}.$$

(135) Gas Constant.
$$R = \frac{M_1 R_1 + M_2 R_2 + M_3 R_3 + \ldots}{M_1 + M_2 + M_3 + \ldots} = \frac{(n_1 + n_2 + n_3 + \ldots) K}{n_1 m_1 + n_2 m_2 + n_3 m_3 + \ldots}.$$

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* R_1 R_2 R_3, etc., in these equations refer to the various gas coefficients of the several gases.

(136) Inner Energy. $M\,U = M_1\,U_1 + M_2\,U_2 + M_3\,U_3 + \ldots.$

(137) Entropy. $M\,S = M_1\,S_1 + M_2\,S_2 + M_3\,S_3 + \ldots.$

(138) Concentration. $c = \dfrac{n_1\,m_1 + n_2\,m_2 + n_3\,m_3 + \ldots}{n_0\,m_0}.$

(139) $$c\,R = \frac{(n_1 + n_2 + n_3 + \ldots)\,K}{n_0\,m_0} = \frac{(n_1 + n_2 + n_3 \ldots)}{n_0}\,\frac{K_0}{m_0}.$$

The Kinetic Theory of Gases for the Atmosphere

The various formulas involving specific heat can all be deduced from the kinetic theory of gases, and it is therefore desirable to have at least approximate values of the constants of the principal gases which are the constituents of the atmosphere. These are arranged in Table 7, so that the formulas from which they are derived suggest by definition the exact meaning of the several terms. It is much better to depend upon formulas for defining constants than upon any extended verbal description for the sake of accuracy and brevity. On the other hand, it is not possible to study any advanced research problem in atmospheric physics without depending upon the several terms in the kinetic theory of gases. In the present status of physics, research is attempting to make out the connection between the theory of mechanical collisions in the molecules of a gas and the corresponding dynamic electric and magnetic forces, but this investigation is incomplete.

It would be very desirable that some international commission should adopt a series of consistent constants for the terms of Table 7, in order that all computations may be made on the same basis. At present there are small variations in the values in consequence of adopting slightly different fundamental constants from which the others are derived. It is probable that sufficient agreement exists among chemists and physicists as to these elementary constants, in order to make this a practical proposition.

TABLE 7

SUMMARY OF FORMULAS AND CONSTANTS FROM THE KINETIC THEORY OF GASES FOR THE ATMOSPHERE

C. G. S. System

Formulas.	Air Mixture	Aqueous Vapor H_2O	Oxygen O_2	Nitrogen N_2	Carbonic Oxide CO	Carbonic Dioxide CO_2	Hydrogen H_2
m = the molecular weight	28.736	17.88	31.76	28.02	27.88	43.76	2.00
$\rho = m\,P/K\,T$ = density	.00129305	.0008046	.0014292	.0012609	.0012546	.0019691	.0000900
$R = K/m$ = gas constant	2870330	4613000	2597000	2943600	2958430	1884830	41240000
$P = \rho\,R\,T$ = Check	This check is complete for each gas.						
$n = \rho/m = P/K\,T$ = atomic density	.000045	Constant for the series.					
$q^2 = 3\,P/\rho$; q = molecule velocity	48486	61466	46119	49100	49223	39290	183779
$\gamma^2 = \frac{8}{3\pi}q^2$; γ = mean velocity	44669	56629	42489	45237	45350	36198	169320
$h = 3/2\,m\,q^2 = n/2\,P$ = constant	Is a constant parameter 2.2204×10^{-11}						
η = coefficient of friction	.000172	.000097	.000191	.000167	.000167	.000145	.000084
$l_{max} = \eta/0.30967\,m\,n\,\gamma$ = path	.00000962	.00000688	.00001018	.00000945	.00000948	.00000657	.00001780
$\nu = \gamma/l$ = number of collisions	4645×10^6	8237×10^6	4175×10^6	4784×10^6	4784×10^6	5510×10^6	9512×10^6
$O = 1/4\sqrt{2}\,l$ = sum of molecular sections	18383	25714	17364	18696	18650	26910	9931
$\bar{v}$ = coefficient of contraction*	.00190	.00282	.00182	.00219	.00186	.00293	.00203
$s = 6\sqrt{2}\,l\bar{v}$ = diameter of a molecule	155×10^{-9}	165×10^{-9}	157×10^{-9}	176×10^{-9}	186×10^{-9}	163×10^{-9}	306×10^{-9}
$\bar{n} = 6\,\bar{v}/\pi s^3$ = number†	97×10^{16}	121×10^{16}	89×10^{16}	77×10^{16}	69×10^{16}	128×10^{16}	13×10^{16}
$l_{max}\sqrt{2}\,\pi s^2\,\bar{n} = 1$, check	This check is complete for each gas.						
$L = 1.667\,\eta\,C_v$ = coefficient conduction‡	484×10^{-7}	582×10^{-7}	496×10^{-7}	483×10^{-7}	484×10^{-7}	403×10^{-7}	3392×10^{-7}
D = coefficient of diffusion	0.1616						
C_p = Specific heat constant pressure	9935000	19813200	9115500	10203500	10232900	9082000	142633333
C_v = Specific heat constant volume	7065000	15196800	6516430	7261333	7274000	7198667	101390700
$R = K/m = C_p - C_v$	2870000	4616400	2599070	2942167	2958900	1883333	41242633
$k = C_p/C_v$	1.40625	1.3038	1.3989	1.4052	1.4068	1.2617	1.4067
C_p in calories	0.2374	0.4734	0.2178	0.2438	0.2445	0.2170	3.4080
C_v in calories	0.1688	0.3631	0.1557	0.1735	0.1738	0.1720	2.4276
R in calories	0.0686	0.1103	0.0621	0.0703	0.0707	0.0450	0.9854
$k = C_p/C_v$	1.4066	1.3038	1.3989	1.4052	1.4068	1.2617	1.4067
C_v by experiment						0.1666	
k by experiment						1.3022	

* $\bar{v}$ is the contraction in passing from the gaseous to the liquid state.

† Electric experiments give larger values of $\bar{n}$.

‡ C_v is in thermal units, but is given below in mechanical units.

This table was computed for the constants,

$K = 82481110.$ Absolute gas constant, dynes/cm.^{2}

$P = 1013235.$ One atmosphere in dynes/cm.^{2}

$\frac{g_0}{A_0} = 41852800.$ Mech. equivalent heat in ergs.

It is very desirable that the International Meteorological Committee should fix standard values throughout the table.

The Temperature and the Temperature Gradients Observed at Different Elevations in the Free Air

The actual temperature of the atmosphere at any point is the resultant of the force of gravitation as balanced by the

TABLE 8

EXAMPLES OF TEMPERATURE AND TEMPERATURE GRADIENTS AT DIFFERENT ELEVATIONS

Station	Lindenburg Apr. 27, 1909 Lat. + 52°		Lindenburg May 5, 1909 Lat. + 52°		Atlantic Ocean Sept. 25, 1907 Lat. + 35° Long. + 36°		Victoria Nyanza Summer, 1908 Lat. 0°	
Height z in meters	T	$\frac{\Delta T}{1000}$	T	$\frac{\Delta T}{1000}$	T	$\frac{\Delta T}{1000}$	T	$\frac{\Delta T}{1000}$
19000	.....	......	.....	.....	......	......	.....	......
18000	.....	......	226.5	+ 3.0	.....	......	190.5	− 6.5
17000	.....	......	223.5	+ 1.0	222.9	+ 2.7	197.1	− 5.5
16000	.....	......	222.5	+ 1.8	220.2	+ 0.1	202.6	− 4.2
15000	.....	......	220.7	+ 1.0	220.1	+ 0.2	206.8	− 4.0
14000	.....	......	219.7	+ 2.8	219.9	− 0.2	210.8	− 5.2
13000	202.7	− 0.8	216.9	+ 5.6	220.1	− 0.2	216.0	− 6.6
12000	203.5	− 6.4	211.3	− 1.7	220.3	− 7.8	222.6	− 8.8
11000	209.9	− 8.8	213.0	− 8.4	228.1	− 7.9	231.4	− 7.5
10000	218.7	− 9.9	221.4	− 9.1	236.0	− 10.0	238.9	− 7.2
9000	228.6	− 11.9	230.5	− 8.8	246.0	− 7.7	246.1	− 4.6
8000	240.5	− 8.8	239.3	− 8.4	253.7	− 7.4	250.7	− 7.3
7000	249.3	− 7.0	247.7	− 7.9	261.1	− 6.3	258.0	− 5.4
6000	256.3	− 6.6	255.6	− 7.3	267.4	− 6.6	263.4	− 5.8
5000	262.9	− 6.2	262.9	− 4.2	274.0	− 4.5	269.2	− 6.5
4000	269.1	− 6.6	267.1	− 5.8	278.5	− 3.4	274.7	− 6.1
3000	275.7	− 5.3	272.9	− 3.9	281.9	− 5.2	280.8	− 7.6
2000	281.0	− 5.5	276.8	− 2.9	287.1	− 3.4	288.4	− 7.8
1000	286.5	− 7.9	279.7	− 1.6	290.5	− 6.0	296.2	......
000	294.4	......	281.3	......	296.5	......	......	......

hydrostatic pressure, the circulation, and the radiation. It is the most important element to be observed, and from it all the other terms can be computed, provided the velocity and the vapor pressure are also given by the observations. In order to have the data in concrete form so that the formulas may become practical, four examples are taken from the observations, at Lindenburg, Germany, in the Tropic North Atlantic Ocean, and at Victoria Nyanza, Africa. Table 8 records the height in meters z, the absolute temperature T, the vertical temperature gradient per 1000 meters $\frac{\Delta T}{1000}$, and Table 9 the relative humidity $R. H.$, and the vapor pressure in millimeters e. The latter is computed by taking from the Tables of vapor pressure in saturated air at given temperatures the dew-point vapor pressure, and multiplying by the relative humidity. The Smithsonian tables have been extended to include approximately the vapor pressure

TABLE 9

EXAMPLES OF THE CORRESPONDING RELATIVE HUMIDITY AND VAPOR PRESSURE AT DIFFERENT ELEVATIONS

Height z	$R. H.$	e	$R. H.$	e	$R. H.$	e	$R. H.$	e
	per cent.	mm.	per cent.	mm.	per cent.	mm.	per cent.	mm.
19000	..	......	..	......	..	......	..	......
18000	..	......	43	0.020	..	......	33	0.000
17000	..	......	43	.004	47	0.014	33	.000
16000	..	......	43	.012	47	.012	33	.001
15000	..	......	44	.011	47	.012	34	.002
14000	..	......	44	.010	47	.012	34	.003
13000	61	0.002	44	.007	47	.012	34	.004
12000	61	.002	43	.004	47	.012	34	.010
11000	61	.043	43	.004	47	.027	34	.029
10000	61	.012	43	.012	47	.063	34	.053
9000	61	.039	43	.033	47	.133	34	.134
8000	62	.135	44	.084	48	.402	35	.218
7000	64	.284	45	.209	51	.843	36	.449
6000	69	.741	47	.473	54	1.544	41	.831
5000	72	1.400	50	.972	56	2.750	50	1.667
4000	67	2.370	51	1.420	58	3.910	73	3.770
3000	67	3.710	42	1.910	66	5.600	69	5.440
2000	70	5.590	41	2.460	65	7.770	60	7.800
1000	63	7.250	60	4.390	72	10.700	57	12.030
000	54	15.620	58	4.730	78	16.770	..	......

at very low temperatures. These values of T and e will be used in illustrating the barometric reduction formulas. By formula

(22) the adiabatic temperature gradient per 1000 meters is -9.869° C., or per 1000 feet -5.415° F. In Table 8 it is seen how widely the actual temperature gradients differ from this value, and it is this circumstance that compels us to reconstruct the entire range of standard thermodynamic formulas, in order to adapt them to practical work in the earth's atmosphere. The slow progress of meteorological physics is due to this difference of gradient more than to any other cause. The temperature gradients are incessantly varying in the atmosphere from large temperature falls, $-\Delta T$, to considerable temperature gains, $+\Delta T$. This change of sign gives rise to the subject of the inversion of temperature of which examples occur often at night near the ground, and usually in the isothermal region.

The values of the vapor pressure below $T = 273^\circ$ are those of the Smithsonian Tables extended.

The Temperature Gradient in a Plateau from the Sea Level to the Surface of the Ground

An important part of barometry is the determination of the temperature gradient within the land mass forming a plateau region, as in the Rocky Mountain district of the United States, by means of which the pressure observed at a station on the surface may be reduced to the sea level, in order to be combined with those stations having low elevations, so as to make a synchronous map of storm conditions for the entire country. This problem is one of considerable difficulty, and it must be solved in accordance with the prevailing local conditions, so that no fixed rules can be given for its treatment. A very extensive reduction for the United States is found in the "Report on Barometry," already mentioned, but its leading principles can be briefly summarized. Having low-level stations in the eastern, central, and Pacific districts, the problem is to connect up the stations on the plateau at different elevations above the sea level, by means of the average temperature gradient within the land mass, which is very different from the gradient above the plateau in the free air. The first step is to construct from the available data approximate temperature gradients, which can be used for short distance reductions, in longitude, in latitude, and on the vertical. Then certain reference vertical lines are chosen, as the intersection of the parallels and meridians for each 5-degree interval, and for the planes 1000 feet apart. To these points are reduced the stations by numerous combinations, so that the same station is reduced to several selected reference points. These points now lie on vertical lines, and each line of temperatures may by plotting be extended downward to sea level. The second step is to draw the sea-level isotherms between the central and the Pacific districts, joining across the plateau region by the most probable curves. The third is to compare by interpolation these horizontal temperatures with those obtained from vertical extension, and by mutual adjustments the two sets may be made to agree harmoniously. This interlocking of a horizontal system with a vertical system is able to produce by mutual checking a very exact agreement between the two sets. In this way the sea-level temperature was found beneath the plateau, and thence the temperature gradients in a vertical direction were computed. These gradients differ greatly from one another in different parts of the plateau, they differ from month to month at the same place, and there is no fixed gradient which can be used at any given station. The station gradients fall into distinct classes in the several parts of the plateau in respect to the yearly variations, and diagrams were constructed to serve for the individual stations. The monthly gradient may be roughly summarized for comparison with the free-air gradients.

The reduction from $\frac{\Delta \text{ T °C.}}{100 \text{ meters}}$ to $\frac{\Delta \text{ T °F.}}{100 \text{ feet}}$ is effected by the factor $\frac{1}{1.82} = 0.55$. This is from $\frac{1}{3.28} \times \frac{1.8}{1} = \frac{1}{1.82} = 0.55$.

This plateau gradient is only 37 per cent. of the free air adiabatic gradient. Extensive reduction barometric tables were constructed for many stations in the United States, which are used in compiling the weather forecast charts, of which further mention will be made.

TABLE 10

THE MEAN MONTHLY TEMPERATURE GRADIENTS IN THE ROCKY MOUNTAIN PLATEAU OF THE UNITED STATES

$\Delta T/100$	Jan.	Feb.	March	Apr.	May	June	
Per 100 ft.	°F. −0.191	°F. −0.174	°F. −0.145	°F. −0.202	°F. −0.194	°F. −0.215	
Per 100 m.	°C. −0.348	°C. −0.317	°C. −0.264	°C. −0.368	°C. −0.353	°C. −0.391	
$\Delta T/100$	July	Aug.	Sept.	Oct.	Nov.	Dec.	Year
Per 100 ft.	°F. −0.175	°F. −0.179	°F −0.168	°F. −0.191	°F. −0.181	°F. −0.187	°F −0.202
Per 100 m.	°C. −0.319	°C. −0.326	°C. −0.306	°C. −0.348	°C. −0.329	°C. −0.340	°C. −0.368

The Integral Mean Temperature and Gradient

In reducing pressures from one level to another it is necessary to know the mean temperature of the actual air column in the free air, or in the hypothetical air column within a plateau, or between a mountain summit and the sea level, or other plane of reference. These are found from the summation of the temperatures at several levels as in Table 8, and dividing the sum by the number of the strata taken.

(140) $$\theta = T_m = \frac{1}{z_1 - z_0}\sum_{z_0}^{z_1} T_z = \frac{1}{z_1 - z_0}\int_{z_0}^{z_1} (T_0 - a\, dz),$$ from (23).

For two strata it is the arithmetical mean,

(141) $T_{10} = \frac{1}{2}(T_1 + T_0)$.

For example, in the balloon ascension, April 27, 1909, sum the T_z through the several strata; also, only the top and bottom of the same thick layer, as indicated.

Layer in meters	$\frac{1}{s}\Sigma T$	$\frac{1}{2}(T_1 + T_0)$
000 to 4000	281.34	281.75
000 to 9000	264.43	260.15
000 to 13000	248.51	243.55

It is not usually sufficient to take the mean value of the top and bottom temperature of a thick layer for the mean temperature of the column, and the error of reduction is dependent upon the discrepancy between T_m and T_{10}. If the gradient is uniform between two strata $T_m = T_{10}$, and the difference vanishes. The length of vertical distance that permits T_{10} to be used depends upon local temperature distributions, and each case must be carefully examined. The same rule applies to the vapor pressure e, and any other meteorological element. The determination of the integral mean with accuracy is one of the hardships of practical meteorology, upon which a large amount of labor is necessarily expended.

The Virtual Temperature T_r.

It is sometimes convenient to combine the actual mean temperature T_m with the expression for the vapor pressure term $0.378\frac{e}{B}$ to form the so-called virtual temperature T_r, by the formula,

$$T_r = T_m \left(1 + 0.378\frac{e}{B}\right) \tag{142}$$

The barometric reduction can then be carried forward as if the dry air and the aqueous vapor were compounded in one gas whose equivalent temperature is T_r.

The Integral Temperature Ratio $\int \frac{dT}{T}$.

The ratio of the change of temperature $d\,T$ to the prevailing temperature T is related to the logarithm in the following useful auxiliary formulas, which are often needed in substitutions.

They are applicable through such strata, thin or thick, as have uniform temperature gradients, whether the temperature increases or diminishes in a vertical direction. They are given for the adiabatic and the non-adiabatic temperature variations.

(143) $$\int \frac{dT_a}{T_a} = \frac{T_a - T_0}{T_{a0}} = \frac{1}{M} \log \frac{T_a}{T_0} = \frac{1}{M} (\log T_a - \log T_0).$$

(144) $$\int \frac{dT_1}{T} = \frac{T_1 - T_0}{T_{10}} = \frac{1}{M} \log \frac{T_1}{T_0} = \frac{1}{M} (\log T_1 - \log T_0).$$

(145) $$\frac{T_1 - T_0}{T_{10}} = \frac{1}{(n-1)M} \log \frac{R_1}{R_0} = \frac{1}{(n-1)M} (\log R_1 - \log R_0).$$

The Temperature Variations and the Specific Heat

It is convenient to make the transfer from the non-adiabatic temperature loss to the adiabatic temperature loss, in connection with the specific heat, by using, as in (13),

$$n_1 = \frac{T_a - T_0}{T_1 - T_0}$$

(146) $$n_1 Cp_a (T_1 - T_0) = Cp_a (T_a - T_0) = - g_0(z_1 - z_0).$$

(147) $$n_1 Cp_{10} (T_1 - T_0) = Cp_{10} (T_a - T_0) = \frac{P_1 - P_0}{\rho_{10}}$$

These formulas will be fully illustrated in a later chapter. It is evident that many combinations can be made by employing the formulas (143) to (147), and they are very practical in developing the formulas.

Transformation of $\frac{P_0}{P} = \frac{B_0}{B} \left(1 + 1.25 \frac{z_1 - z_0}{R}\right)$.

The introduction of the plateau effect upon gravity in (69) has its parallel in the effect upon the barometric pressure, which is similarly modified. We have for both cases,

(148) $$g_0 = g_z \left(1 + 1.25 \frac{z_1 - z_0}{R}\right).$$

(149) $$\frac{P_0}{P} = \frac{B_0}{B} \left(1 + 1.25 \frac{z_1 - z_0}{R}\right).$$

Since $1 + 1.25 \dfrac{z_1 - z_0}{R}$ is a small variation from unity, the general formula is applicable,

(150) Com. log $(1 + x) = M\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots\right)$.

Passing to common logarithms (149) becomes on neglecting the powers above x,

$$(151)\quad \log \frac{P_0}{P} = \log \frac{B_0}{B} + \log\left(1 + 1.25 \frac{z_1 - z_0}{R}\right),$$

$$= \log \frac{B_0}{B} + \frac{M\,1.25}{R}(z_1 - z_0).$$

It will be found in the barometric formula that,

(152) $z_1 - z_0 = K \log \dfrac{P_0}{P}$ (approx.) $= K \log \dfrac{B_0}{B}$ (approx.). Hence,

$$(153)\quad \log \frac{P_0}{P} = \log \frac{B_0}{B} + 1.25 \frac{M\,K}{R} \log \frac{B_0}{B},$$

$$= \log \frac{B_0}{B}\left(1 + 1.25 \frac{M\,K}{R}\right),$$

$$= \log \frac{B_0}{B}(1 + 0.00157) = \log \frac{B_0}{B}(1 + y).$$

The General Barometric Formula

The several auxiliary formulas now deduced make it very simple to derive the barometric reduction formula connecting together the heights $(z_1 \,.\, z_0)$ and the mercurial pressures (B_1, B_0). From (41) the differential pressure is,

(154) $- d\,P = \rho \,.\, g\,d\,z.$ Divide by P,

(155) $- \dfrac{d\,P}{P} = \dfrac{\rho}{P} g\,d\,z.$ Divide by $\dfrac{\rho}{P}$,

(156) $- \dfrac{d\,P}{P} \,.\, \dfrac{P}{\rho} = g \,.\, d\,z.$ Substitute from (75),

$$(157)\quad - \frac{d\,P}{P} \,.\, \frac{P_0}{\rho_0}\frac{T}{T_0}\left(1 + 0.378 \frac{e}{B_0}\right) = g_{\phi z}\,d\,z.$$

Substitute P_0 from (49) and $g_{\phi z}$ from (63), (64),

(158) $$-\frac{dP}{P} \cdot \frac{B_0\,\rho_m\,g_0}{\rho_0}\frac{T}{T_0}\left(1 + 0.378\frac{e}{B_0}\right) = g_0\,(1 - 0.0026\cos 2\phi)\left(1 - \frac{2z}{R}\right)dz.$$

Pass to common logarithms by the factor $\frac{1}{M}$ and integrate,

(159) $$\log\frac{P_0}{P} \cdot \frac{B_0\,\rho_m}{M\,\rho_0} \cdot \frac{T_m}{T_0}\left(1 + 0.378\frac{e}{B_0}\right)_m = (1 - 0.0026\cos 2\phi)\left(1 - \frac{z_1 + z_0}{R}\right)(z_1 - z_0).$$

The last gravity form is from (67). The constant

(160) $$K = \frac{B_0\,\rho_m}{M\,\rho_0} = 18400\text{ (Metric), and } K_1 = 60367.7\text{ (English).}$$

T_m is the mean temperature of the column (140), and the integral mean of $\left(1 + 0.378\frac{e}{B_0}\right)$ is accomplished by the observations along the column, or by integrating from the surface by (82) and (83). Substituting (153)

(161) $$\log\frac{B_0}{B}\,(1 + 0.00157)\,K\,(1 + 0.00367\,\theta)\left(1 + 0.378\frac{e}{B_0}\right)_m$$

Barometer. Plateau Effect. Const. Temperature. Vapor Pressure.

$$(1 + 0.0026\cos 2\phi)\left(1 + \frac{z_1 + z_0}{R}\right) = z_1 - z_0.$$

Gravity in Latitude. Gravity in Height. Height.

Substituting the numerical values and combining,

(162) $$\log B_0 = \log B + \frac{z_1 - z_0}{18429 + 67.6\,\theta^\circ\text{C.} + 0.003\,z}\left(1 - 0.378\frac{e}{B_0}\right)(1 - 0.0026\cos 2\phi).\quad\text{Metric.}$$

(163) $$\log B_0 = \log B + \frac{z_1 - z_0}{56517 + 123.3\,\theta^\circ\text{F.} + 0.003\,z}\left(1 - 0.378\frac{e}{B_0}\right)(1 - 0.0026\cos 2\phi).\quad\text{English.}$$

These can be expressed in the general form,

$$(164)\quad \log B_0 = \log B + m\,(1 - \beta)\,(1 - \gamma)$$
$$= \log B + m - m\,\beta - m\,\gamma.$$

In view of the uncertainty attaching to our knowledge at any time of the distribution of the vapor pressure in the air column, it is desirable to keep the term $m\,\beta$ separate as a correction to the difference between the logarithms. Similarly the gravity term is retained by itself because in many computations it is small and can be neglected. Complete reduction tables are given in the "Report on the International Cloud Observations" for the metric system, Tables 91, 92, 93, and in the "Report on the Barometry of the United States, Canada, and the West Indies" for the English system, Tables 13 to 21. From these logarithm tables many forms of numerical tables without logarithms can be constructed for special purposes.

Corrections to the Barometer

The mercurial barometer requires several corrections before the pressure can be used in practice.

1. *Correction to the Standard Temperature.* The instrument is constructed of parts whose coefficients of expansion with changes of temperature are not the same, as for the mercury and the brass scale. Adopting the notation,

t = the temperature of the attached thermometer.

t_m = the standard temperature of mercury, 0° C., 32° F.

t_s = the standard temperature of the brass scale, 0° C., 62° F.

m = the coefficient of expansion of mercury, 0.0001818 per degree Centigrade, 0.0001010 per degree Fahrenheit.

n = the coefficient of expansion of brass, 0.0000184 per degree Centigrade, 0.0000102 per degree Fahrenheit.

The accepted formulas are as follows:

$$(165)\quad B_n - B = -B\,\frac{(m-n)\,t}{1 + m\,t} \text{ for } B_n \text{ and } B \text{ in millimeters.}$$

$$(166)\quad B_n - B = -B\,\frac{m\,(t - 32^\circ) - n\,(t - 62^\circ)}{1 + m\,(t - 32^\circ)} \text{ for } B_n \text{ and } B \text{ in inches.}$$

The English form reduces to,

$$(167)\quad B_n - B = -B\frac{t - 28.63}{10978 + 1.112\,t}.$$

The necessary reduction tables are found in nearly all compilations of Meteorological Tables.

2. *Correction to the Standard Gravity*, g_{45}. This is the gravity variation in latitude as given in (63), from which is obtained,

$$(168)\quad B_{45} - B_\phi = -B_\phi\left(1 - \frac{g_\phi}{g_{45}}\right) = -B_\phi\,0.00260\ \cos 2\phi.$$

The temperature and gravity corrections are applied as instrumental corrections to the actual barometric reading at a given hour.

3. *Correction to a Standard Barometer or Patron.* Each barometer as an instrument has certain minor deficiencies which cannot be readily analyzed, and in order to make a number of barometers homogeneous, so as to give strictly comparable pressures, it is necessary that they be severally standardized by comparison with an adopted normal or patron barometer. The Kew barometer is used for many standard comparisons, and each weather service keeps its own standard which has been carefully compared with the Kew instrument. Within each national service the barometers are compared, and an instrumental correction is given for each barometer before sending to a station. Sometimes these corrections hold steadily for long intervals, and sometimes they change suddenly and erratically. Whenever there are local removals, or whenever a barometer is cleaned, its correction must again be determined. Frequent inspections and comparisons with a portable secondary standard are necessary if a homogeneous series of pressures is to be secured. It is not possible to be too painstaking in respect of the interbarometric corrections.

4. *The Station or Removal Correction.* If it happens that at a given station there are any removals of the barometer from one office to another, as so frequently happens in large cities, and the elevation is thereby changed from time to time, it is necessary to adopt a standard elevation for the station and reduce the

series of readings taken at any other height to this level, which will persist as the adopted station elevation from the beginning to the end of the service. When the change in height is considerable these corrections depend upon the temperature in the course of the year. A correction card for instrumental and station removal errors should accompany each barometer, and be carefully recorded as part of the history of the instrument. In preparing homogeneous tables of pressure for use in solar physics and other cosmical problems, it is indispensable that all barometer readings should be carefully treated in this manner. The homogeneous system for the United States has thus been prepared by the author to cover the barometric pressure, the temperature, the vapor pressure, and the precipitation from the year 1871 to date, and the published data of the Weather Bureau are all on that basis. Similar homogeneous data are being prepared for Argentina and for other countries.

5. *The Correction from the Surface Temperature* (t) *to the Mean Temperature of the Air Column* (θ) *in Barometric Reductions.* It is obviously so difficult to determine the relation of the surface temperature t to the mean air column temperature within a land mass θ, as from a station on a plateau to the sea level, that a special study was made of this subject in order to facilitate a prompt reduction of the observed pressure to the corresponding sea-level pressure. These are needed for transmission by telegraph to a central office where the daily weather forecast charts are constructed. Unfortunately there is no simple rule connecting t and θ, and in many cases the difference $\theta - t$ is very variable. Reduction tables are first computed with the adopted elevation H, and a series of assumed values of θ for several barometric pressures in steps of 0.10 inch. Then the relation between t and θ having been found, the surface temperature is used as the argument for the table in place of θ. The practical value of t taken in the United States, where the observations are made at 8 A.M. and 8 P.M. daily, is the mean of the current dry-bulb temperature and that taken twelve hours before. This gives a fair temperature average for the day, and it tends to eliminate some of the local effects of passing storms. It has

been found to work well in the practice of ten years. In order to illustrate the differences between t and θ in the course of the year, as the temperatures change from summer to winter, a few examples are extracted from Table 53 of the Barometry Report, where the heights are in feet, and temperatures are Fahrenheit.

TABLE 11

RELATION BETWEEN THE SURFACE TEMPERATURE t AND THE MEAN θ

Boise, Idaho 2739		Salt Lake City, Utah 4366		Independence, Cal. 3910		Helena, Mont. 4110		Pike's Peak, Colo. 14111		Battleford, Canada 1608	
t	θ	t	θ	t	θ	t	θ	t	θ	t	θ
−42	−40	..	..	..	..	−48	−40	−61	−40	..	..
−32	−30	..	..	..	..	−37	−30	−49	−30	−40	−30
−23	−20	−24	−20	..	..	−26	−20	−35	−20	−29	−20
−13	−10	−14	−10	..	..	−15	−10	−23	−10	−17	−10
− 3	− 0	− 5	0	− 3	0	− 4	0	−12	0	− 6	0
7	10	5	10	6	10	7	10	− 2	10	5	10
17	20	14	20	15	20	20	20	5	20	16	20
27	30	24	30	24	30	29	30	10	30	27	30
37	40	34	40	35	40	38	40	16	40	38	40
47	50	44	50	48	50	46	50	22	50	49	50
57	60	55	60	60	60	55	60	30	60	59	60
67	70	66	70	71	70	66	70	38	70	69	70
77	80	76	80	81	80	76	80	47	80	79	80
87	90	86	90	92	90	86	90	56	90	89	90
97	100	96	100	102	100	96	100	..	..	99	100

Similarly, the relations between the surface t and the mean free-air temperature, or the mean plateau temperature, θ, have been prepared for reductions to the sea-level plane, the 3,500-foot level, and the 10,000-foot planes for over 200 stations, so that synchronous charts can be constructed on each of these three planes simultaneously. Such charts were prepared for one year, in part by telegram and in part by card reports, so that the pressure charts could be studied on the sea level, on the mean-plateau level, and in the two-mile level. These comparisons were so suggestive and instructive in respect of the progress of storms and the areas of precipitation as to make them of great value in practical forecasts of weather conditions. The trend of the upper-level isobars shows clearly the course of the storm track for 24 to 36 hours, whereas the sea-level isobars have very little evidence of this kind. This is because the closed isobars on the sea level have generally opened up into sweeping curves on the two-mile level. Similarly, the rain areas are indicated by the region of most oblique crossing of the lower with the upper isobars. There is a great future for meteorology in the use of these upper level charts.

6. *The Plateau Correction* $C \Delta\theta H$. An extensive discussion of the reduced pressures on the sea-level plane showed a series of plateau differences depending upon a station constant $C = 0.001$ usually, $\Delta \theta =$ the departure of the monthly from the annual θ, and $H =$ the height of the station in feet in units of 1,000 feet, so that $\Delta B = C. \Delta \theta. H$. This plateau correction was computed and applied to all the plateau stations of the United States. It seems to take account of the effect of the land mass in the course of the year upon the temperature distribution, which is very complex in its action.

7. *The Local Correction* ΔA. After the corrections above mentioned have been applied, there are still a few stations which require a small correction ΔA to make them harmonize with the pressure system surrounding them. The cause is still obscure and is very local, possibly due to the wind action near the office.

8. *The Local Vapor Pressure Correction.* The prevailing relative humidity and the corresponding vapor pressure are approximate functions of the temperature in each locality, so that an approximate value of the correction to the barometer due to the presence of the aqueous vapor can be found for each station and applied along with the other corrections.

9. *The Station Pressure Reduction Charts.* It should be noted that all the barometric corrections have been made in terms of the surface temperature so that this t and the barometer reading B, when corrected for the several instrumental errors, become the arguments for the reduction to any plane. In practical work, instead of corrections from the station to the sea level or other plane of reference being furnished to the several stations, there have been prepared, for the arguments (t, B), the reduced sea-level pressure at once in a sufficiently expanded form of tables to

permit of quick and accurate interpolation; similar tables were provided for the 3,500-foot and the 10,000-foot planes. With these auxiliary station tables the reduced pressures are promptly obtained, and transformed into the telegraph cipher code for transmission to other offices. In this way a large number of stations in the United States receive in several telegraph circuits, by interchange of messages, the necessary data for all map construction. Making the observations at 8 A.M., 75th meridian time in all districts, for example, the data are received, recorded, interpreted as forecasts, and usually retransmitted to all parts of the country within two hours, or by 10 o'clock.

Examples of the Barometric Reduction Tables

"Barometry Report," U. S. W. B., 1900–01, Tables 13–21. "International Cloud Report," U. S. W. B., 1898–99, Tables 91–93.

In order to illustrate the barometric formulas in practice, the example of Santa Fé, N. M., is here given.

TABLE 12

REDUCTION TO THE SEA LEVEL BY THE *m*-TABLES IN LOGARITHMS

SANTA FÉ, NEW MEXICO

Height = 7013 feet. Longitude, 105° 57′ Latitude, 35° 41′

Arguments	Jan.	April	July	Oct.	Year
Station........ B	23.180 in.	23.177 in.	23.362 in.	23.294 in.	23.248 in.
Temperature...θ	32.0° F.	54.3° F.	79.0° F.	58.0° F.	56.0° F.
Vapor Pressure e_0	0.145 in.	0.199 in.	0.574 in.	0.294 in.	0.274 in.
Logarithm log B.	1.36511	1.36506	1.36851	1.36725	1.36638
Table 17.......m	+.11594	+.11090	+.10581	+.11011	+.11054
Table 19C, $-\beta m$	−.00016	−.00022	−.00061	−.00033	−.00031
Table 20 $-\gamma m$	−.00011	−.00010	−.00010	−.00010	−.00010
Sum...... log B_0	1.48078	1.47564	1.47361	1.47693	1.47651
Sea level.... B_0	30.254	29.898	29.759	29.987	29.958
Table 19A Arg I.	.0018	.0025	.0073	.0037	.0035
Table 19B Arg. II	.0014	.0020	.0058	.0030	.0028
Reduction $B_0 - B$	7.074	6.721	6.397	6.693	6.710

SECOND FORM OF TABLE, NUMERICAL

Table 21, Sec. I. .	7.028	6.762	6.487	6.719	6.742
Table 21, Sec. II.	+.064	−.018	−.041	+.004	−.003
$-\beta m - \gamma m$	−.018	−.022	−.049	−.030	−.028
Reduction......	7.074	6.722	6.397	6.693	6.711

The station pressure B is already corrected for the temperature, gravity, instrumental and removal errors.

For the argument $\frac{1}{2}(t_{8p} + t_{8a})$ or $\frac{1}{2}(t_{8a} + t_{8p})$, take θ.

For the arguments $(B.\ \theta)$, in Table 17, take m.

For the arguments $(B_0.\ e_0)$ in Table 19 A, take Arg. I (below).

For the arguments (Arg. I, H), in Table 19 B, take Arg. II (below).

For the arguments (Arg. II, m), Table 19 C, take $-\beta m$.

For the arguments (ϕ, m), in Table 20, take $-\gamma m$.

Numerical Form

For the arguments (H, B = 30 inches), in Table 21, I, take first reduction.

For the arguments H, B_0 (approx.), in Table 21, II, proceed by trials.

Interpolate the correction for humidity from Table 21, III.

Interpolate the correction for gravity from Table 21, IV.

These two methods work very rapidly after a little practice, and the reductions are valid to the 0.001 inch of pressure. In order to illustrate the method of reduction for the plateau the following example is given in Table 13.

The assumed station pressure B has the four station corrections applied. The body of the reduction table was computed for assumed values of θ, which correspond with certain surface temperatures t, computed from the consecutive 8 o'clock pairs as observed. This t becomes then the practical argument for station reductions. The mean annual θ for Santa Fe was taken 63° F., and $\Delta\theta$ at any time is the variation (θ − 63°). The

TABLE 13

Santa Fé, New Mexico

Reduction of Pressure to the Sea Level, the 3,500- and 10,000-Foot Planes

I.—Reduction to sea level

Elevation, 7,013 feet. Longitude, 105° 57′. Latitude, 35° 41′

Temp.		Correction for			22.40	22.60	22.80	23.00	23.20	23.40	23.60	23.80	24.00	24.20
t	θ	$C.\Delta\theta.H$	e	ΔA	Reduction $B_0 - B$ from m-Table.									
−27	−20	−.50	....	....	+7.79	7.86	7.93	8.00	8.07	8.14	8.21	8.28	8.35	8.42
−16	−10	−.44	...	....	.59	. 66	.73	7.80	7.86	7.93	7.99	8.06	8.3	.20
− 5	0	−.38	...	...	.40	.46	.53	.60	.66	.73	.79	7.86	7.93	8.00
5	10	−.32	.00	.00	:22	.28	.34	.41	.47	.54	.60	.67	.73	7.80
16	20	−.26	−.01	.00	7.05	7.11	.17	.24	.30	.36	.42	.49	.55	.61
26	30	−.20	−.01	.00	6.88	6.94	7.00	7.07	7.13	.19	.25	.31	.37	.43
35	40	−.14	−.01	.00	.72	.78	6.84	6.90	6.96	7.02	7.08	7.14	.20	.26
43	50	−.08	−.01	.00	.57	.63	.69	.75	.81	6.87	6.93	6.99	7.05	7.11
50	60	−.02	−.01	.00	.43	.49	.55	.60	.66	.72	.77	.83	6.89	6.95
58	70	+.04	−.02	.00	.29	.35	.41	.46	.52	.58	.63	.68	.74	.80
67	80	+.10	−.03	.00	.16	.22	.28	.33	.39	.44	.49	.54	.60	.66
77	90	+.16	−.03	.00	6.04	6.09	.15	.20	.26	.31	.36	.41	.47	.52
88	100	+.22	−.03	.00	5.91	5.97	6.02	6.07	6.13	6.18	6.23	6.28	6.34	6.39

Date	Jan.	Feb.	Mch.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
5	28.9	30.6	37:0	44.2	52.7	62.0	67.4	67.0	62.0	53.2	41.7	33.1
15	27.9	32.0	39.5	46.6	55.7	65.1	68.5	66.3	59.9	49.8	37.6	30.8
25	29.3	34.5	41.9	49.6	58.8	66.2	67.8	64.2	56.5	45.7	35.3	29.8

Note.—Δ A and $C.\Delta\theta.H$ have been united.

II.—Reduction to the 3,500-foot plane

Temp.		Correction for			22.40	22.60	22.80	23.00	23.20	23.40	23.60	23.80	24.00	24.20
t	θ_1	$C.\Delta\theta.H$	e	ΔA	Reduction $B_1 - B$ from m-Table									
−24	−20	−.27	...	...	+3.61	3.65	3.68	3.71	3.75	3.78	3.81	3.84	3.87	3.90
−14	−10	−.24	...	...	.52	.56	.59	.62	.66	.69	.72	.75	.78	.81
− 4	0	−.21	...	...	.44	.47	.50	.53	.57	.60	.63	.66	.69	.72
6	10	−.18	...	...	.36	.39	.42	.45	.48	.51	.54	.57	.60	.63
16	20	−.15	...	...	.29	.32	.35	.38	.41	.44	.46	.49	.52	.55
26	30	−.12	...	...	.22	.25	.28	.30	.33	.36	.39	.42	.45	.47
36	40	−.09	...	...	.15	.18	.21	.23	.26	.29	.32	.35	.38	.40
45	50	−.06	...	...	.08	.11	.14	.16	.19	.22	.25	.28	.31	.33
55	60	−.03	.00	.00	3.02	3.05	.08	.10	.13	.16	.18	.21	.24	.26
64	70	.00	−.01	.00	2.96	2.99	3.02	3.04	.07	.10	.12	.15	.17	.19
74	80	+.03	−.01	.00	.90	.93	2.96	2.98	3.01	3.04	.06	.09	.11	.13
84	90	+.06	−.02	.00	.84	.87	.90	.92	2.95	2.98	3.00	3.03	3.05	.07
94	100	+.09	−.02	.00	2.79	2.82	2.84	2.86	2.89	2.92	2.94	2.97	2.99	3.01

III.—Reduction to the 10,000-foot plane

Temp.		Correction for			22.40	22.60	22.80	23.00	23.20	23.40	23.60	23.80	24.00	24.20
t	θ_2	$C.\Delta\theta.H$	e	ΔA	Reduction $B_2 - B$ from m-Table.									
−20	−20	−.21	...	...	−2.67	2.70	2.73	2.75	2.78	2.80	2.82	2.85	2.87	2.89
−10	−10	−.18	...	...	− .61	.64	.67	.69	.72	.74	.76	.79	.81	.83
1	0	−.15	...	...	− .56	.59	.61	.63	.66	.68	.70	.73	.75	.77
12	10	−.12	...	...	− .51	.54	.56	.58	.61	.63	.65	.67	.69	.71
23	20	−.09	...	...	− .46	.49	.51	.53	.56	.58	.60	.62	.64	.66
34	30	−.06	...	...	− .41	.44	.46	.48	.51	.53	.55	.57	.59	.61
45	40	−.03	.00	.00	− .37	.39	.41	.43	.46	.48	.50	.52	.54	.56
56	50	−.01	+.01	.00	− .32	.34	.36	.38	.41	.43	.45	.47	.49	.51
67	60	+.01	+.01	.00	− .28	.30	.32	.34	.37	.39	.41	.43	.45	.47
78	70	+.03	+.01	.00	− .24	.26	.28	.30	.32	.34	.36	.38	.40	.42
89	80	+.06	+.02	.00	− .20	.22	.24	.26	.28	.30	.32	.34	.36	.38
100	90	+.09	+.02	.00	−2.16	2.18	2.20	2.22	2.24	2.26	2.28	2.30	2.32	2.34

value of C for Santa Fé happens to be 0.00086 and H is taken in units of 1,000 feet, 7.01. Hence, for the sea level,

for $\theta =$ − 20,	$C.\Delta\theta.H = 0.00086 \times$	$(-83) \times 7.01 =$	− 0.50
0		(− 63)	− 0.38
+ 20		(− 43)	− 0.26
+ 40		(− 23)	− 0.14
+ 60		(− 3)	− 0.02
+ 80		(+ 17)	+ 0.10
+ 100		(+ 37)	+ 0.22

The plateau stations always seem to require such a correction in order to make a harmonious network of pressures with the surrounding low-level stations. It is easier to make this correction in the form given above, rather than attempt to trace out its effect upon the mean temperature θ, as related to the surface temperature t. The entire subject needs a fuller theoretical discussion if possible. The vapor pressure correction e is the mean value as for the argument surface t, and suffices for these station reduction tables up to the 0.01 inch. The final station reductions to the sea level were made for the arguments (t. B), and applied to the assumed values of B, so that for the same arguments (t. B), the value of B_0 is immediately read by an easy interpolation.

Similar reductions were made for the 3,500-foot plane, and the 10,000-foot plane. They were checked by reduction from the station to 3,500 feet, to sea level, from sea level to 10,000 feet, and thence in a circuit back to the station pressure. This was done for all the numerous plateau stations in the United States.

CHAPTER II

Thermodynamic Meteorology

General Formulas for the Computation of P, ρ, R, from the Observed Temperatures T in a Free Non-Adiabatic Atmosphere

It is easily seen from the discussion of the barometer how many complexities this instrument introduces in practice, on account of the series of corrections, and by reason of the system of units employed, which separates the data from all other thermodynamic terms occurring in meteorology. There is need, then, of developing another system of reduction, by which it may be possible to pass from the temperatures T, observed in the free air up to great heights, to the corresponding pressures P, densities ρ, and gas coefficients R, so that the general law $P = \rho R T$ shall continuously be satisfied throughout the atmosphere. If the mercurial barometer is needed on the surface to give a base for vertical reductions, it is not practical to carry it to heights on kites and balloons. The aneroid may be used to check the resulting computed pressures, but not to give the actual pressure for the dependent formulas. Fortunately, there is a simple and comprehensive set of formulas for this purpose, which will now be developed.

For any temperature vertical gradient a, the temperature T at the height z above T_0 is,

(169) $T = T_0 - a\,z$, so that, $d\,T = -\,a\,d\,z$, and $d\,z = -\dfrac{d\,T}{a}$.

The differential equation for pressure variations with the height is from (41),

(170) $-\,d\,P = \rho\,g_0\,d\,z = -\dfrac{1}{a}\,\rho\,g_0\,d\,T$, by substituting $d\,z$.

From the Boyle-Gay Lussac Law, $P = \rho\,R_0\,T$, by division, we obtain, since by (24) $\dfrac{g_0}{a\,R_0} = \dfrac{n\,k}{k-1}$,

(171) $\dfrac{d\,P}{P} = \dfrac{g_0}{a\,R_0}\dfrac{d\,T}{T} = \dfrac{n\,k}{k-1}\dfrac{d\,T}{T}$.

Passing to logarithms and to limits, this gives,

$$(172) \quad \frac{P_1}{P_0} = \left(\frac{T_1}{T_0}\right)^{\frac{n\,k}{k-1}}.$$

Having observed T_1 and T_0 on two levels, at the vertical distance apart $z_1 - z_0$, the pressure P_1 can be computed from the pressure P_0 on the lower level. We proceed to determine the density ρ_1, and gas coefficient R_1, which correspond with $P_1 = \rho_1 R_1 T_1$ on the z_1-level, when $P_0 = \rho_0 R_0 T_0$ is given on the z_0-level. By successive stages from the surface the same formulas will arrive at any altitude where the temperature T is known. From two successive levels, we have the ratio,

$$(173) \quad \frac{P_1}{P_0} = \frac{\rho_1 R_1 T_1}{\rho_0 R_0 T_0}, \text{ and by transforming,}$$

$$(174) \quad \frac{\rho_1 R_1}{\rho_0 R_0} = \frac{P_1 T_0}{P_0 T_1} = \left(\frac{T_1}{T_0}\right)^{\frac{n\,k}{k-1} - 1} = \left(\frac{T_1}{T_0}\right)^{\frac{n}{k-1} + (n-1)}.$$

At this point the entire treatment of thermodynamic meteorology diverges. If the gas coefficient is taken constant, $R_1 = R_0$, and

$$(175) \quad \frac{\rho_1}{\rho_0} = \left(\frac{T_1}{T_0}\right)^{\frac{n\,k}{k-1} - 1} = \left(\frac{T_1}{T_0}\right)^{\frac{g_0}{R_0 a} - 1}.$$

For example, V. Bjerknes in equation B, page 51, "Dynamic Meteorology and Hydrography," Carnegie Institution of Washington, 1910, uses this form for the ratio ρ_1/ρ_0, since in his system of units $g_0 = 1$. This is the common way of treating the matter, but it is easy to see that this derivation of the non-adiabatic densities from the well-known adiabatic equation is inconsistent with the analogue of the pressures in (172), which simply multiplies the exponent $\frac{k}{k-1}$ by n, so that for the densities the exponent should be $\frac{n}{k-1}$.

Proceeding in the second way it is obvious that, preserving the same treatment for density as for pressure, we should take,

$$(176) \quad \frac{\rho_1}{\rho_0} = \left(\frac{T_1}{T_0}\right)^{\frac{n}{k-1}}.$$

$$(177) \quad \frac{R_1}{R_0} = \left(\frac{T_1}{T_0}\right)^{(n-1)}.$$

In order to check these results by (172) and (173),

$$(178) \quad \frac{P_1}{P_0} = \left(\frac{T_1}{T_0}\right)^{\frac{nk}{k-1}} = \left(\frac{T_1}{T_0}\right)^{\frac{n}{k-1} + (n-1) + 1},$$

which is correct. This process makes R a variable in the existing non-adiabatic atmosphere, so that the air is not distributed by gravitation like an adiabatically expanding gas, in which there is no circulation and no change of heat contents by radiation and absorption from level to level. On the contrary, the observations prove that usually there is circulation and radiation going on to preserve the gravitation equilibrium with the existing pressure variations or gradients. As stated, the entire system of thermodynamics takes on a new form through the fact that the specific heat must also be a variable along with the gas coefficient.

$$(2) \qquad C_p = \frac{k}{k-1} R.$$

We shall return to explain the consequences of this fundamental property of the atmosphere, which is in reality a gaseous mixture of rapidly varying thermodynamic capacities, in consequence of the effect of the absorption of solar radiation and the emission of atmospheric radiation in various ways.

The Adiabatic Equations

The correlative adiabatic equations follow at once by putting $n = 1$, and $a = a_0$,

$$(179) \qquad \text{Pressure } \frac{P_1}{P_0} = \left(\frac{T_1}{T_0}\right)^{\frac{k}{k-1}}.$$

$$(180) \qquad \text{Density } \frac{\rho_1}{\rho_0} = \left(\frac{T_1}{T_0}\right)^{\frac{1}{k-1}}.$$

(181) Gas constant $R_1 = R_0$.

The Working Non-Adiabatic Equations

(182) Pressure, $\log P_1 - \log P_0 = \frac{n\,k}{k-1} (\log T_1 - \log T_0).$

(183) Density, $\log \rho_1 - \log \rho_0 = \frac{n}{k-1} (\log T_1 - \log T_0).$

(184) Gas coefficient, $\log R_1 - \log R_0 = (n-1)\ (\log T_1 - \log T_0).$

These equations were published in the *Monthly Weather Review*, March, 1906, (38), (39), (40), and they have been illustrated by numerous applications to balloon and kite ascensions with excellent results, up to great altitudes, as 20,000 meters. The following example shows the method of arranging the computation so as to proceed from level to level, the computed P_1, ρ_1, R_1 of one becoming P_0, ρ_0, R_0 for that next above it. The example is taken at random from our computations, some of the results being compiled in Bulletin No. 3, Argentine Meteorological Office, 1913. The constants are taken from Table 3 in the (M. K. S.) system. The surface values of P_0, ρ_0, R_0, T_0 are assumed to conform to the adiabatic system, while P_1, ρ_1, R_1, T_1 above the surface are computed by the non-adiabatic system.

At the height z = 116 meters the density is computed from the adiabatic formula,

$$\rho = \frac{P}{T\,R_0}.$$

Also, $P = g_0\,\rho_m\,B_0$, where B_0 is in meters. It will be noted that the check is complete. It may be stated that the observed values of B_0 at Lindenburg are usually about 1 mm. higher than the computed values B_c. This constitutes a criterion upon the adjustment of the aneroid, which, in ascending, lags in registration and records a pressure corresponding to a lower level than that assumed for the temperature T at the height z.

TABLE 14. EXAMPLE OF THE NON-ADIABATIC COMPUTATIONS FOR P_1, ρ_1, R_1

Lindenburg, Germany, April 27, 1909

Height z	116	500	1000	1500	2000	2500	3000	4000	5000
Difference $z_1 - z_0$	....	384	500	500	500	500	500	1000	1000
Temperature t	21.4	17.1	13.5	10.3	8.0	5.7	2.7	−3.9	−10.1
$T_1 - T_0$	....	−4.3	−3.6	−3.2	−2.3	−2.3	−3.0	−6.6	−6.2
$\log (T_a - T_0)$	....	−0.57862	−0.69326	....	....	....	....	−0.99429	....
$\log (T_1 - T_0)$	....	−0.63347	−0.55630	−0.50515	−0.36173	−0.36173	−0.47712	−0.81954	−0.79239
$\log n = \frac{T_a - T_0}{T_1 - T_0}$	....	9.94515	0.13696	0.18811	0.33153	0.33153	0.21614	0.17475	0.20190
n	....	0.88136	1.3707	1.5421	2.1455	2.1455	1.6449	1.4954	1.5919
$\log \frac{n k}{k-1}$	....	0.48442	0.67623	0.72738	0.87080	0.87080	0.75541	0.71402	0.74117
T	294.4	290.1	286.5	283.3	281.0	278.7	275.7	269.1	262.9
$\log T$	2.46894	2.46255	2.45712	2.45225	2.44871	2.44514	2.44044	2.42991	2.41979
$\log T_1 - \log T_0$	....	−.00639	−.00543	−.00487	−.00354	−.00357	−.00470	−0.1053	−.01012
$\log (\log T_1 - \log T_0)$	....	−7.80550	−7.73480	−7.68753	−7.54900	−7.55267	−7.67210	−8.02243	−8.00518
$\log \frac{n k}{k-1} (\log T_1 - \log T_0)$	....	−8.28992	−8.41103	−8.41491	−8.41980	−8.42347	−8.42751	−8.73645	−8.74635
$\log P_1 - \log P_0$	....	−.01950	−.02576	−.02600	−.02629	−.02651	−.02676	−.05451	−.05576
$\log P$	4.99775	4.97825	4.95249	4.92649	4.90020	4.87369	4.84693	4.79242	4.73666
P	99482	95115	89638	84428	79470	74763	70296	62004	54532
$\log B$	9.87286	9.85336	9.82760	9.80160	9.77531	9.74880	9.72204	9.66753	9.61177
Computed B_c	0.7462	0.7135	0.6724	0.6333	0.5961	0.5608	0.5273	0.4651	0.4090
Observed B_0	....	0.7140	0.6730	0.6340	0.5970	0.5620	0.5290	0.4670	0.4110
$B_c - B_0$	....	−.0005	−.0006	−.0007	−.0009	−.0012	−.0017	−.0019	−.0020
$\log \frac{n}{k-1}$	....	0.33636	0.52817	0.57932	0.72274	0.72274	0.60735	0 56596	0.59311

$\log \frac{n}{k-1} (\log T_1 - \log T_0)$	....	−8.14186	−8.26297	−8.26685	−8.27174	−8.27541	−8.27945	−8.58839	−8.59829
$\log \rho_1 - \log \rho_0$	....	−.01386	−.01832	−.01849	−.01870	−.01885	−.01903	−.03876	−.03965
$\log \rho$	0.07088	0.05702	0.03870	0.02021	0.00151	9.98266	9.96363	9.92487	9.88522
ρ	1.1773	1.1403	1.0932	1.0476	1.0035	0.9609	0.9197	0.8411	0.7678
$\rho_a{}^* = P/R_a T$	1.1773	1.1423	1.0900	1.0383	0.9853	0.9346	0.8883	0.8028	0.7227
$\rho - \rho_a$	.0000	− .0020	+ .0032	+.0093	+.0182	+.0263	+.0314	+.0383	+.0451
$\log (n-1)$	....	−9.07423	9.56902	9.73408	0.05899	0.05899	9.80949	9.69496	9.77225
$\log (n-1)(\log T_1 - \log T_0)$	....	6.87973	−7.30382	−7.42161	−7.60799	−7.61166	−7.48159	−7.71739	−7.77743
$\log R_1 - \log R_0$	....	+.00076	−.00201	−.00264	−.00406	−.00409	−.00303	−.00522	−.00599
$\log R$	2.45793	2.45869	2.45668	2.45404	2.44998	2.44589	2.44286	2.43764	2.43165
R	287.03	287.53	286.21	284.47	281.82	279.18	277.24	273.93	270.18
Check $\log P = \log (T \rho R)$	4.99775	4.97826	4.95250	4.92650	4.90020	4.87369	4.84693	4.79242	4.73666

Equations. (182) $\log P_1 - \log P_0 = \frac{n\,k}{k-1} (\log T_1 - \log T_0)$.

(183) $\log \rho_1 - \log \rho_0 = \frac{n}{k-1} (\log T_1 - \log T_0)$.

(184) $\log R_1 - \log R_0 = (n-1)(\log T_1 - \log T_0)$.

(173) $\log P = \log T + \log \rho + \log R$.

Constants. $\log \frac{k}{k-1} = 0.53927$. $\log \frac{1}{k-1} = 0.39121$. $\log g_0 \rho_m = 5.12489$

Comparison of ρ (non-adiabatic) and ρ_a (adiabatic).

ρ (non-adiabatic)	1.1773	1.1403	1.0932	1.0476	1.0035	0.9609	0.9197	0.8411	0.7678
ρ_a (adiabatic)	1.1773	1.1423	1.0900	1.0383	0.9853	0.9346	0.8883	0.8028	0.7227

* The adiabatic density ρ_a is computed for R_a = constant 287.03, and it is generally smaller than the true density ρ. All discussions depending on ρ_a, R_a, in the atmosphere are incorrect.

TABLE 15

COMPARISON OF THE PRESSURES COMPUTED BY THE NON-ADIABATIC FORMULA AND THOSE OBSERVED AT GREAT HEIGHTS

Heights in Meters	Lindenburg May 5, 1909 Lat. +52°			Lindenburg July 27, 1908 Lat. +52°			Atlantic Ocean Sept. 9, 1907 Lat. +26°			Atlantic Ocean June 19, 1906 Lat. −2°		
z	B_c	B_0	B_c-B_0	B_c	B_0	B_c-B_0	B_c	B_0	B_c-B_0	B_c	B_0	B_c-B_0
19000	....	....	....	....	....	....	....	....	......	....	....	....
18000	....	....	....	.0589	.0600	−.0011	....	....	......	....	....	....
17000	.0679	.0670	+.0009	.0692	.0690	+ 2	.0742	.0740	+.0002	....	....	....
16000	.0794	.0790	+ 4	.0808	.0790	+ 18	.0861	.0860	+ 1	....	....	....
15000	.0927	.0930	− 3	.0945	.0950	− 5	.1004	.1010	− 6	.1021	.1020	+.0001
14000	.1082	.1090	− 8	.1101	.1110	− 9	.1177	.1170	+ 7	.1203	.1200	+ 3
13000	.1266	.1270	− 4	.1289	.1300	− 11	.1381	.1380	+ 1	.1409	.1410	− 1
12000	.1484	.1490	− 6	.1512	.1520	− 8	.1615	.1610	+ 5	.1641	.1640	+ 1
11000	.1744	.1750	− 6	.1772	.1780	− 8	.1881	.1880	+ 1	.1902	.1900	+ 2
10000	.2041	.2050	− 9	.2067	.2080	− 13	.2179	.2180	− 1	.2191	.2190	+ 1
9000	.2374	.2390	− 16	.2399	.2410	− 11	.2509	.2510	− 1	.2512	.2510	+ 2
8000	.2746	.2760	− 14	.2769	.2780	− 11	.2875	.2880	− 5	.2870	.2870	0
7000	.3159	.3180	− 21	.3181	.3190	− 9	.3281	.3290	− 9	.3268	.3260	+ 8
6000	.3619	.3630	− 11	.3639	.3650	− 11	.3732	.3740	− 8	.3709	.3710	− 1
5000	.4128	.4130	− 2	.4149	.4160	− 11	.4233	.4240	− 7	.4200	.4200	0
4000	.4696	.4700	− 4	.4716	.4720	− 4	.4789	.4800	− 11	.4747	.4750	− 3
3000	.5330	.5340	− 10	.5344	.5340	+ 4	.5402	.5410	− 8	.5356	.5350	+ 6
2000	.6035	.6040	− 5	.6040	.6040	0	.6077	.6060	+ 17	.6031	.6030	+ 1
1000	.6823	.6840	− 17	.6806	.6810	− 4	.6821	.6820	+ 1	.6783	.6780	+ 3
Surface	.7599	.7599	0	.7551	.7551	0	.7640	.7640	0	.7610	.7610	0

The differences between B_c and B_0 are probably due to an assignment of the temperature to a slightly erroneous height,* owing to the movement of the balloon ahead of the record of the barograph and thermograph, which requires a correction for lag. The pressure recorded by the aneroid is for the mixture of dry air and aqueous vapor, so that by (75) for the same height, where $T = T_0$ and $\rho = \rho_0$,

$$(185) \qquad P_0 = P\left(1 - 0.378\frac{e}{P}\right)$$

$$B_0 = B\left(1 - 0.378\frac{e}{B}\right)$$

where P or B is the dry air pressure, and P_0 or B_0 the pressure in the mixture, e being the vapor pressure in the

* The heights have been read from the aneroid record and are not corrected for the supposed lag; but the error is less for T than for P because T changes more slowly than P.

same system of units. The connection between (182) and (159) is such that they can easily be shown to be identical, after the action of the mercurial barometer has been made to equal that of an aneroid.

The Variable Values of $n = \dfrac{a_0}{a}$

The introduction of n into the adiabatic formulas converts them into the non-adiabatic formulas, and at the same time adds circulation and radiation to a static atmosphere. Hence, by (13),

$$(186)\qquad n = \frac{a_0}{a} = -\frac{T_a - T_0}{z_1 - z_0} \bigg/ -\frac{T_1 - T_0}{z_1 - z_0} = \frac{-(T_a - T_0)}{-(T_1 - T_0)}$$

marks the natural transition from static to dynamic and thermodynamic meteorology. It is important, therefore, to understand the full significance of the ratio between the adiabatic and the non-adiabatic temperature gradients. Since $(T_a - T_0)$

TABLE 16

EXAMPLES OF THE VALUE OF $n = \frac{a_0}{a}$

Height z	Lindenburg May 5, 1909	Lindenburg July 27, 1908	Atlantic Ocean Sept. 9, 1907
18000	......	− 3.6553	......
17000	+1.0966	−16.4489	− 2.5972
16000	−5.4830	− 6.1684	− 0.9399
15000	−9.8694	−49.3467	− 2.9027
14000	−3.5248	− 6.1684	+ 4.9347
13000	−1.7624	− 2.5972	+ 1.9739
12000	+5.8055	+ 2.0999	+ 1.3337
11000	+1.1749	+ 1.1611	+ 0.9676
10000	+1.0846	+ 1.2986	+ 1.1611
9000	+1.1215	+ 1.1344	+ 1.1611
8000	+1.1749	+ 1.1749	+ 1.3901
6000	+1.3520	+ 1.6179	+ 1.3901
4000	+1.7016	+ 1.8980	+ 1.2653
2000	+3.5248	+ 1.2986	+ 2.5972
1000	−4.9347	+12.3366	−12.3367

and $(T_1 - T_0)$ are usually each negative with an increase in elevation, n is generally a positive quantity, but it becomes negative whenever there is an inversion of temperature, or temperature increase with the height, as near the surface of the ground in the early morning, or in the isothermal layer at great heights. If $T_1 = T_a$, $n = 1$, and the gradient is adiabatic; if $T_1 = T_0$, $n = \infty$ and there is no temperature change with the height; if $T_1 > T_0$, n is negative, and if T_1 is only a little greater than T_0 for the change in elevation $z_1 - z_0$, n will be a large negative quantity. Table 16 gives a few examples of the values of n.

Table 16 indicates the wide range through which n passes in practical reductions, and it is easily seen how valueless the formulas become for meteorological discussions where n is assumed to be unity, as is commonly the procedure. Furthermore, since the value of n must always be carried to the fourth decimal it has not seemed worth while to construct general reduction tables, because they would be very extensive or require complex interpolation.

The Differentiation of (172)

Since n is a variable in equation (172), we proceed to differentiate it for P, T, n, variables.

(187) $$\log\left(\frac{P}{P_0}\right) = \frac{n\,k}{k-1}\log\left(\frac{T}{T_0}\right).$$ Differentiate,

(188) $$d\left(\frac{P}{P_0}\right)\Big/\frac{P}{P_0} = \frac{n\,k}{k-1}\,d\left(\frac{T}{T_0}\right)\Big/\frac{T}{T_0} + \log\frac{T}{T_0}\cdot\frac{k}{k-1}\,\frac{d\,n}{M}.$$

(189) $$\frac{d\,P}{P} = \frac{n\,k}{k-1}\,\frac{d\,T}{T} + \frac{1}{M}\,\frac{k}{k-1}\log T\,d\,n.$$

Substitute $P = \rho\,R\,T$ and $Cp = R\,\frac{k}{k-1}$,

(190) $$\frac{d\,P}{\rho} = n\,Cp\,d\,T + \frac{1}{M}\,Cp\,T\log T\,d\,n.$$

Take the integral between limits for the mean ρ_{10},

(191) $$\frac{P_1 - P_0}{\rho_{10}} = n_1\,Cp_{10}\,(T_1 - T_0) + \frac{1}{M}\,Cp_a\,T_{10}\log\frac{T_1}{T_0}\,(n_1 - n_0).$$

By (144), (146), (147), using the mean values Cp_{10},

$$(192) \quad \frac{P_1 - P_0}{\rho_{10}} = n_1 Cp_{10} (T_1 - T_0) + (n_1 - n_0) Cp_a (T_1 - T_0).$$

$$= n_1 Cp_{10} (T_1 - T_0) + n_1 Cp_a (T_1 - T_0) - n_0 Cp_a (T_1 - T_0)$$

$$= Cp_{10} (T_a - T_0) + Cp_a (T_a - T_0) - Cp_a (T_a - T_0)$$

$$= Cp_a (T_a - T_0) - (Cp_a - Cp_{10}) (T_a - T_0).$$

Since ρ and Cp are variable in the stratum $(z_1 - z_0)$, the mean values are ρ_{10} and Cp_{10}, while n_1 continues constant within the stratum, which must not be taken too thick to allow this approximation to hold true. The result is twofold. First,

$$(193) \quad \frac{P_1 - P_0}{\rho_{10}} = Cp_{10} (T_a - T_0) = n_1 Cp_{10} (T_1 - T_0).$$

The adiabatic system, on the other hand, gives,

$$(194) \quad \frac{P_a - P_0}{\rho_{a0}} = Cp_a (T_a - T_0) = - g (z_1 - z_0).$$

Hence, the difference between the two systems is,

$$(195) \quad \frac{P_a - P_0}{\rho_{a0}} - \frac{P_1 - P_0}{\rho_{10}} = (Cp_a - Cp_{10}) (T_a - T_0).$$

From the common dynamic equation for pressure and velocity, which will be proved in a later section, and adding a term for the dynamic energy of radiation heat, we obtain by substitution in (192) the working and fundamental equation, as the second result,

$$(196) \quad g_0 (z_1 - z_0) = - \frac{P_1 - P_0}{\rho_{10}} - \tfrac{1}{2} (q_1^2 - q_0^2) - (Q_1 - Q_0).$$

These have already been quoted in (21), (25), (26), (27).

From (192), using the last form, we find,

$$(197) \quad g_0 (z_1 - z_0) = - \frac{P_1 - P_0}{\rho_{10}} - (Cp_a - Cp_{10}) (T_a - T_0),$$

and by comparison of the last terms (196), (197),

$$(198) \quad - (Q_1 - Q_0) - \tfrac{1}{2} (q_1^2 - q_0^2) = - (Cp_a - Cp_{10}) (T_a - T_0).$$

(199) $\quad -(Q_1 - Q_0) = -(Cp_a - Cp_{10})(T_a - T_0) + \frac{1}{2}(q_1^2 - q_0^2).$

That is to say, the variation from the true adiabatic system is due to the kinetic energy of heat $(Q_1 - Q_0)$, and circulation $\frac{1}{2}(q_1^2 - q_0^2)$ for the unit mass, which is equivalent to the variation of the mean specific heat of the stratum from the adiabatic specific heat, $Cp_a = 993.5787$ in Table 3, multiplied by the change in temperature between the bottom T_0 and top T_a of the layer $z_1 - z_0$. Unfortunately, there seems to be no way to separate $(Q_1 - Q_0)$ from $\frac{1}{2}(q_1^2 - q_0^2)$ through the specific heat, except by using the direct observations of the velocity, and then computing $(Q_1 - Q_0)$ by means of (199). Those observatories which record P, T, the pressure and the temperature, but not the $R. H.$ and q, the relative humidity and velocity of motion of the stratum, cannot enter upon any problem in circulation or radiation in the atmosphere.

From equation (196) we obtain,

(200) $\quad -\frac{dP}{\rho} = g\,dz + q\,dq + dQ$, and,

(201) $\quad -dP = g\,\rho\,dz + \rho\,q\,dq + \rho\,dQ.$

From the equation (190) is derived,

(202) $\quad -dP = -\rho\,n\,Cp\,dT - \rho\frac{1}{M}Cp\,T\log T.\,dn,$

and by means of (146), (144), (21), this becomes,

(203) $\quad -dP = g\,\rho\,dz - \rho\,\frac{1}{M}\,Cp\,T\log T.\,dn.$

The usual adiabatic pressure variation $-dP = g\,\rho\,dz$, as in (170), is converted into the non-adiabatic form with circulation and radiation by subtracting $\rho\,\frac{1}{M}\,Cp\,T\log T\,dn$, which includes the differentials first for heat, and second for circulation.

If equation (203) is treated in the same manner as (170) we shall obtain by substitutions,

(204) $\quad \left(\frac{T}{T_0}\right)_{ad.} - \left(\frac{T}{T_0}\right)_{non\text{-}ad.} = (a_0 - a)\,\frac{z_1 - z_0}{T_0} = \left(1 - \frac{1}{n}\right)\frac{a_0}{T_0}\,\frac{z_1 - z_0}{T_0},$

which is the difference between the temperature ratios in the adiabatic and the non-adiabatic systems.

TABLE 17

EXAMPLE OF THE NON-ADIABATIC COMPUTATIONS, $-(Q_1 - Q_0)$ BY THE FORMULAS (199) AND (196)

Lindenburg, April 27, 1909

Height z	116	500	1000	1500	2000	2500	3000	4000	5000
$\log Cp = R \frac{k}{k-1}$	2.99720	2.99796	2.99595	2.99331	2.98925	2.98516	2.98213	2.97691	2.97092
Cp	993.58	995.32	990.72	984.72	975.55	966.40	959.68	948.22	935.24
$\frac{1}{2}(Cp_1 + Cp_0) = Cp_{10}$	....	994.45	993.02	987.72	980.14	970.98	963.04	953.95	941.73
Cp_a	993.58	993.58	993.58	993.58	993.58	993.58	993.58	993.58	993.58
$Cp_a - Cp_{10}$	....	−0.87	+0.56	+5.86	+13.44	+22.60	+30.54	+39.63	+51.85
$T_a - T_0$	....	−3.79	−4.94	−4.94	−4.94	−4.94	−4.94	−9.87	−9.87
$-(Cp_a - Cp_{10})(T_a - T_0)$	....	−3.3	+2.8	+28.9	+66.4	+111.6	+150.9	+391.1	+511.8
Velocity q in m/sec	9.0	14.0	13.4	13.0	14.8	17.5	20.0	22.4	26.8
q^2	81.0	196.0	179.6	169.0	219.0	306.2	400.0	501.8	718.2
$q_1^2 - q_0^2$	....	+115.0	−16.4	−10.6	+50.0	+87.2	+93.8	+101.8	+216.4
$-\frac{1}{2}(q_1^2 - q_0^2)$	(3)	−57.5	+8.2	+5.3	−25.0	−43.6	−46.9	−50.9	−108.2
(199) $-(Q_1 - Q_0)$	(2)	+54.2	−5.4	+23.6	+91.4	+155.2	+197.8	+442.0	+620.0
$-(P_1 - P_0)$	....	4367.	5477.	5210.	4958.	4707.	4467.	8292.	7472.
$\frac{1}{2}(\rho_1 + \rho_0) = \rho_{10}$	....	1.1588	1.1168	1.0704	1.0256	0.9822	0.9403	0.8804	0.8044
$\log -(P_1 - P_0)$	....	3.64018	3.73854	3.71684	3.69531	3.67274	3.65002	3.91866	3.87344
$\log \rho_{10}$	....	0.06401	0.04797	0.02954	0.01098	9.99220	9.97327	9.94468	9.90547
		3.57617	3.69057	3.68730	3.68433	3.68054	3.67675	3.97398	3.96797
$-\frac{(P_1 - P_0)}{\rho_{10}}$	(1)	3768.5	4904.2	4867.4	4834.2	4792.2	4750.6	9418.5	9289.0

TABLE 17—*Continued*

Sum (1) + (2) + (3)	....	3765.2	4907.0	4896.3	4900.6	4903.8	4901.5	9809.6	9800.8
$g_0 (z_1 - z_0)$	....	3765.5	4903.0	4903.0	4903.0	4903.0	4903.0	9806.0	9806.0
Check	....	+0.3	−4.0	+6.7	+2.4	−0.8	+1.5	−3.6	+5.2

Equations (2) $Cp = R \frac{k}{k-1}$.　　$Cp_{10} = \frac{1}{2}(Cp_1 + Cp_0)$.

(198) $-(Cp_a - Cp_{10})(T_a - T_0) = -\frac{1}{2}(q_1^2 - q_0^2) - (Q_1 - Q_0) = -(Q_1 - Q_0)'$.

(199) $-(Q_1 - Q_0) = -(Cp_a - Cp_{10})(T_a - T_0) + \frac{1}{2}(q_1^2 - q_0^2)$.

(196) $g_0 (z_1 - z_0) = -\frac{P_1 - P_0}{\rho_{10}} - \frac{1}{2}(q_1^2 - q_0^2) - (Q_1 - Q_0)$.

The algebraic signs must be carefully observed.

From these examples of the computations it is easy to check the elementary formulas.

(146) $-Cp_a(T_a - T_0) = g_0(z_1 - z_0)$....	3765.5	4903.0	4903.0	4903.0	4903.0	4903.0	9806.0	9806.0
(146) $-n_1 Cp_a(T_1 - T_0) = g_0(z_1 - z_0)$...	3765.5	4903.0	4903.0	4903.0	4903.0	4903.0	9806.0	9806.0
(147) $Cp_{10}(T_a - T_0) = \frac{P_1 - P_0}{\rho_{10}}$..........	3768.8	4906.3	4874.1	4836.7	4791.4	4752.3	9414.8	9294.4
(147) $n_1 Cp_{10}(T_1 - T_0) = Cp_{10}(T_a - T_0)$	3768.8	4900.3	4874.1	4836.7	4791.4	4752.3	9414.8	9294.4
(144) $T_1 - T_0 = T_{10}\frac{1}{M}\log\frac{T_1}{T_0}$.........	−4.3	−3.6	−3.2	−2.3	−2.3	−3.0	−6.6	−6.2
(145) $T_1 - T_0 = \frac{1}{(n-1)M} T_{10}\log\frac{R_1}{R_0}$..	−4.3	−3.6	−3.2	−2.3	−2.3	−3.0	−6.6	−6.2

Evidently there are very many other combinations of these fundamental formulas that can be formed and checked.

There are a few critical remarks that can now be made to advantage, in view of the principles that have been assumed in many important meteorological papers.

(1) By (201) and (203) it is not proper to assume that the variation of the pressure in a vertical direction is proportional to the mass variation, $-dP = g\,dm$, because this excludes the circulation and the radiation. It is a contradiction in terms to seek for solutions of radiation problems under this limitation.

(2) By (198), (199), if it is assumed that R and Cp are constants and the change of temperature is adiabatic, there can be no circulation and no radiation. It is a contradiction in terms to seek for solutions of radiation problems under this assumption.

(3) By (193) the integral $\int \frac{dP}{\rho}$ depends upon a varying Cp. It follows that if P. T. are observed by the instruments, if R_0 is taken constant in the Boyle-Gay Lussac Law, and if ρ is then computed directly from $P = \rho R_0 T$, this value in the integral $\int \frac{dP}{\rho}$ will always lead to fictitious results, because the circulation and the radiation are excluded from the discussion.

(4) In the integration of (190), in strata where n is constant, as they can be made by taking the layer thin enough, it is proper to use the mean values of the several terms, ρ_{10}, Cp_{10}, T_{10},

$$\int \frac{dP}{\rho} = \frac{P_1 - P}{\rho_{10}},\ \rho_{10} = \tfrac{1}{2}(\rho_1 + \rho_0).$$

$$\int n\,Cp\,dT = n\,Cp_{10}(T_1 - T_0),\ Cp_{10} = \tfrac{1}{2}(Cp_1 + Cp_0).$$

$$\int \frac{1}{M}\,Cp\,T \log \frac{T}{T_0}\,dn = \frac{1}{M}\,Cp_a\,T_{10} \log \frac{T_1}{T_0}(n_1 - n_0),$$

since $Cp = Cp_a$ and $n_0 = 1$ in this term.

(5) It has been customary to evaluate the equations of motion in the atmosphere while omitting the heat term $-(Q_1 - Q_0)$ in (196). The result is that it is impossible to balance the other three terms, so that the great problem of the relation of the circulation to the observed pressures in the general and the local circulations has been insoluble. In order to exhibit more fully

the relation of these four terms, two examples of balloon ascensions to great heights are added.

TABLE 18

THE EVALUATION OF $-(Q_1 - Q_0)$ IN BALLOON ASCENSIONS (196)

Station	Lindenburg, May 5, 1909					Atlantic Ocean, Sept. 9, 1907			
z	$g(z_1 - z_0)$ *	$-\frac{P_1 - P_0}{\rho_{10}}$	$-\frac{1}{2}(q_1^2 - q_0^2)$	$-(Q_1 - Q_0)$	Δ	$-\frac{P_1 - P_0}{\rho_{10}}$	$-\frac{1}{2}(q_1^2 - q_0^2)$	$-(Q_1 - Q_0)$	Δ
17000	9806.0	6437.6	+ 3.6	3358.6	+ 6.2	6755.8	−26.8	3065.9	+11.1
16000	9806.0	6629.2	+ 5.6	3169.2	+ 2.0	7301.2	+ 6.4	2496.1	+2.3
15000	9806.0	6963.7	+11.2	2817.1	+14.0	7886.4	+38.9	1885.6	− 4.9
14000	9806.0	7357.0	+ 9.2	2433.5	+ 6.3	8288.6	+104.8	1424.2	−11.6
13000	9806.0	7851.0	+ 5.6	1939.7	+ 9.7	8508.6	+53.0	1228.5	+15.9
12000	9806.0	8300.4	+ 1.7	1499.8	+ 4.1	8666.2	−89.3	1227.9	+ 1.2
11000	9806.0	8489.0	− 0.9	1313.9	+ 4.0	8706.6	−33.2	1130.2	+ 2.4
10000	9806.0	8530.6	+ 0.9	1268.8	+ 5.7	8721.0	− 1.4	1079.9	+ 6.5
9000	9806.0	8565.0	+ 8.1	1227.5	+ 5.4	8769.4	+ 4.2	1025.7	+ 6.7
8000	9806.0	8615.8	+13.2	1176.9	+ 0.1	8845.2	+30.4	928.1	+ 2.3
7000	9806.0	8668.8	+16.1	1114.1	+ 7.0	8958.0	−54.3	895.0	+ 7.3
6000	9806.0	8750.2	+ 8.8	1044.0	+ 3.0	9074.0	−53.8	777.6	+ 8.2
5000	9806.0	8885.0	+ 5.2	909.2	+ 6.6	9222.0	+18.3	568.1	− 2.4
4000	9806.0	9051.5	+ 5.4	745.6	+ 3.5	9340.5	−24.6	485.9	+ 4.2
3000	4903.0	4589.3	+ 7.0	312.3	− 5.6	4706.0	− 2.2	204.1	− 4.9
2500	4903.0	4636.4	+11.7	257.5	− 2.6	4725.8	− 6.2	179.2	+ 4.2
2000	4903.0	4689.1	− 5.9	217.3	+ 2.5	4764.9	− 4.5	135.7	+ 6.9
1500	4903.0	4738.9	−22.1	174.4	+11.8	4815.6	− 0.8	87.6	+ 0.6
1000	4903.0	4813.8	−32.5	104.4	+17.3	4846.6	+13.7	9.9	+32.8
500	3765.5	3753.8	−11.5	19.5	+ 3.7	4865.7	−10.7	0.9	− 1.9

* More exactly g should diminish with height.

It is easy to see that in the higher levels $(Q_1 - Q_0)$ is a dominant term, and that there is no possibility of a balance between the other three taken by themselves.. There is a delicate interaction of the four terms, such that the circulation seeks to adjust the pressure and the radiation to the demands of gravity. The minor errors Δ are to be chiefly ascribed to the fact that the observed temperatures are not quite correct at the height z. In many cases there are pair values of $+\Delta$ and $-\Delta$ easily adjusted on this hypothesis.

The Two Laws of Thermodynamics

Before proceeding further it is necessary to summarize the two fundamental laws of thermodynamics, the first being that of the conservation of energy, and the second being that of the decrease of entropy or increase of expenditure in a non-conservative system. The first law is defined as follows: In a conservative system, from which no heat escapes and into which no heat is received, the sum of all the changes in the energy, whether large or small, remains constant. In a system in communication with the outside world the amount of the energy gained or lost by it is equal to that delivered to or received from the outside world. The second law may be described in several ways. Heat cannot by itself pass out of a colder into a warmer body (Clausius). It is impossible to construct a periodically acting machine which does nothing else than raise a weight in expending work and cooling a reservoir of heat (Planck). There exists in nature a quantity which always changes itself only in the same direction in all the variations which take place in nature; this is the entropy. It is in nowise possible to diminish the entropy of a system of bodies without there being left changes in other bodies. If such changes do not remain, then the entropy of a system can continue the same. Every physical or chemical change takes place in such a way that entropy either decreases or remains the same, but the outside world tends continually toward maximum entropy. It is necessary and sufficient for the equilibrium of a separate structure that for all possible changes in the state of the structure, the changes of entropy be zero or negative (Weinstein).

The First Law of Thermodynamics

The following series of formulas are available in the adiabatic and non-adiabatic systems using the proper values of $P. \rho. R. T.$

(205) $P = \rho R T. \qquad \rho = \frac{P}{R T}.$

(206) $P v = R T. \qquad v = \frac{R T}{P}.$

(207) $d T = \frac{P d v + v d P}{R}.$

(208) $d v = \frac{R}{P} d T - \frac{R T}{P} \cdot \frac{d P}{P} = v \frac{d T}{T} - v \frac{d P}{P}.$

(209) $\frac{dv}{v} = \frac{dT}{T} - \frac{dP}{P}.$

Referring to the series of equations 84–118 for the definition of the terms, we have generally,

(210) *Inner Energy.* $U = H + J = Q - W.$

(211) $U' = Cv \int T . \rho d v = \frac{Cv}{R} \int P . d v.$

(212) $d U = d Q - d W = T d S - d W = T d S - P d v.$

(213) $d U = \frac{dQ}{dT} d T + \frac{dQ}{dv} d v - P d v.$

(214) $d U = Cv\, d T + \frac{R T}{v} d v - P d v.$

(215) $d U = Cp\, d T - P d v.$

(216) $d U = Cv\, d T = (Cp - R)\, d T.$

(217) *External Work.* $W = (K) + V = J + H + V.$

(218) $d W = d Q - d U.$

(219) $d W = T d S - d U.$

(220) $d W = d (T S - U) - S d T.$

(221) $d W = - d F - S d T.$

(222) $d W = P d v.$

(223) $d W = R d T - \frac{dP}{\rho}.$

(224) *Heat Energy.* $Q = W + U.$

(225) $Q = W + H + J = H + K.$

(226) $Q = [(K) + V]$ external $+ [H + J]$ internal $+ (R)$ friction.

(227) $d Q = \frac{dQ}{dT} \cdot d T + \frac{dQ}{dv} \cdot d v = d W + d U.$

(228) $d Q = Cv\, d T + \frac{R T}{v} \cdot d v = Cv\, d T + P d v.$

(229 $d Q = Cv\, d T + R d T - v d P = Cp\, d T - R T \frac{dP}{P}.$

(230) $$dQ = Cv \cdot \frac{v\,dP + P\,dv}{R} + P\,dv.$$

(231) $$dQ = T\,dS.$$

From (218) in heat units, we obtain,

(232) *First Law.* $A\,dW = dQ - A\,dU$, and by (222),

(233) $$A\frac{d\,(P\,dv)}{dT\,dv} = \frac{d}{dT}\left(\frac{dQ}{dv}\right) - \frac{d}{dv}\left(\frac{A\,dU}{dT}\right).$$

(234) $$A\frac{dP}{dT} = \frac{d\,A\,P}{dT} - \frac{d\,Cv}{dv}.$$

(235) $$\frac{d\,Cv}{dv} = \frac{d\left(\frac{RT}{v} - P\right)}{dT}.$$

(236) $-Q_1 = -U_1 - W_1$, Heat expended outward (negative).
$+Q_2 = +U_2 + W_2$, Heat received inward (positive).

(237) $\Delta Q = Q_2 - Q_1 = (U_2 - U_1) + (W_2 - W_1)$, Resultant heat energy.

Among the definitions we have,

V = the external gravity potential acting inwards, together with the centrifugal force of the earth's rotation at the angular velocity $\omega_0 = \frac{d\beta}{dt}$, at the perpendicular distance ϖ from the axis,

(238) $$V = -\int g\,dz + \int \tfrac{1}{2}\,(\varpi\,\omega_0)^2 dm.$$

(239) (K) = the kinetic energy of motion of the mass m with the velocity q. $(K) = \frac{1}{2}\,m\,q^2 = H + J$.

(240) F = the free energy or the thermodynamic potential at a constant volume. $F = U - TS$.

(241) U = The bound energy. $U - F = TS$.

(242) ϕ = the thermodynamic potential at a constant pressure, $F + Pv$. $\phi = U - TS + Pv$.
$\phi = U - TS + RT$.

Fundamental Equations and Definitions

It is convenient to have for ready reference the fundamental equations and several definitions of thermodynamic processes, though they cannot be further developed in this connection. It is the purpose of this treatise to prepare such data for meteorology as can be admitted into the large group of well-known equations which have been heretofore inapplicable in the atmosphere for lack of the necessary correct values of $P. \rho. R. T.$

There are several variables: $P.v.T.\ Q.W.U.S.R.$

(243) $$dU = A^1 dQ - dW = A^1 T dS - P dv.$$
$$dW = P dv. \qquad dQ = T dS.$$

(244) $$A^1 dS = \frac{dU + P dv}{T}. \qquad dv = \frac{A^1 T dS - dU}{P}.$$
$$dF = -A^1 S dT - P dv.$$

(245) $$dT = -\frac{dF + P dv}{A^1 S}. \qquad dv = -\frac{dF + A^1 S dT}{P}.$$
$$d\phi = -A^1 S dT + v dP.$$

Adiabatic Processes. $dQ = 0$ and $dS = 0$.

Adiabatic. $dQ = 0$ signifies no gain of heat from the outside and no loss of heat to the outside from the system.

Isentropic. $dS = 0$ signifies that the entropy remains constant.

Isodynamic Processes. $dU = 0$ and $dT = 0$.

Isodynamic. $dU = 0$. The inner energy remains constant.

Isothermal. $dT = 0$. The temperature remains constant.

Isoenergetic Processes. $dW = 0$, $dP = 0$, $dv = 0$, $d\rho = 0$.

Isometric. $dW = 0$. The expenditure of external work is the same.

Isobaric. $dP = 0$. The pressure remains unchanged.

Isochoric. $dv = 0$. The volume is constant during the process.

Isopyknic. $d\rho = 0$. The density is constant during the process.

Isopiestic Processes. $dP = 0$.

Isopiestic. $dP = 0$. The pressure is constant while the other variables change.

Isoelastic Processes. $dR = 0$.

Isoelastic. $dR = 0$. In ideal gases the gas coefficient is constant.

Evaluation of dQ in Terms of $P. v. T.$ through the Entropy S. Taking the following three pairs of variables, they lead to the definition of the specific, latent, and expansion heats.

(246) Variables $(v. T)$. $dQ = T\left(\frac{\partial S}{\partial T}\right)_v dT + T\left(\frac{\partial S}{\partial v}\right)_T dv = Cv\,dT + C_T\,dv.$

(247) Variables $(P. T)$. $dQ = T\left(\frac{\partial S}{\partial T}\right)_P dT + T\left(\frac{\partial S}{\partial P}\right)_T dP = Cp\,dT + \gamma_T\,dP.$

(248) Variables $(P. v)$. $dQ = T\left(\frac{\partial S}{\partial P}\right)_v dT + T\left(\frac{\partial S}{\partial v}\right)_P dv = \gamma_v\,dP + \gamma_p\,dv.$

Hence by comparison the definitions become,

Specific Heats.	*Latent Heats.*	*Expansion Heats.*
(249) $Cp = T\left(\frac{\partial S}{\partial T}\right)_P$.	(250) $C_T = T\left(\frac{\partial S}{\partial v}\right)_T$.	(251) $\gamma_p = T\left(\frac{\partial S}{\partial v}\right)_P$.
(252) $Cv = T\left(\frac{\partial S}{\partial T}\right)_v$.	(253) $\gamma_T = T\left(\frac{\partial S}{\partial P}\right)_T$.	(254) $\gamma_v = T\left(\frac{\partial S}{\partial P}\right)_v$.
These occur in radiation.	These occur in evaporation.	These occur in convection.

The subscript indicates the term which remains constant.

Evaluation of dQ in Terms of $P. v. T.$ through the Inner Energy U.

(255) Variables $(v. T)$. $dQ = A\left(\frac{\partial U}{\partial T}\right)_v dT + A\left[\left(\frac{\partial U}{\partial v}\right)_T + P\right] dv.$

(256) Variables $(P.T)$. $dQ = A\left[\left(\frac{\partial U}{\partial T}\right)_P + P\frac{\partial v}{\partial T}\right]dT +$

$$A\left[\left(\frac{\partial U}{\partial P}\right)_T + P\frac{\partial v}{\partial P}\right]dP.$$

(257) Variables $(P.v)$. $dQ = A\left(\frac{\partial U}{\partial P}\right)_v dP + \left[\left(\frac{\partial U}{\partial v}\right)_P + P\right]dv.$

Specific Heat

(258) $Cp = \left(\frac{\partial Q}{\partial T}\right)_P = A\left[\left(\frac{\partial U}{\partial T}\right)_P + P\frac{\partial v}{\partial T}\right].$

Latent Heat

(259) $C_T = \left(\frac{\partial Q}{\partial v}\right)_T = A\left[\left(\frac{\partial U}{\partial v}\right)_T + P\right].$

Expansion Heat

(260) $\gamma_P = \left(\frac{\partial Q}{\partial v}\right)_P = A\left[\left(\frac{\partial U}{\partial v}\right)_P + P\right].$

Specific Heat

(261) $Cv = \left(\frac{\partial Q}{\partial T}\right)_v = A\left(\frac{\partial U}{\partial T}\right)_v$

Latent Heat

(262) $\gamma_T = \left(\frac{\partial Q}{\partial P}\right)_T = A\left[\left(\frac{\partial U}{\partial P}\right)_T + P\frac{\partial v}{\partial P}\right].$

Expansion Heat

(263) $\gamma_v = \left(\frac{\partial Q}{\partial P}\right)_v = A\left(\frac{\partial U}{\partial P}\right)_v.$

By intercomparisons and substitutions very numerous equations can be constructed. Compare Weinstein's "Thermodynamik." Those for entropy are,

(264) $S_{vT} = S_0 + Cv \log T + A\,R \log v.$

(265) $S_{TP} = S_0 + Cp \log T - A\,R \log P.$

The Second Law of Thermodynamics

This is derived from equation (231).

(266) *Second Law.* $dS = \frac{dQ}{T}.$

This gives rise to two processes in nature, the reversible, in which after a series of transformations the original state is reached, and the irreversible process, in which the original state is permanently lost.

For the reversible process, $d\,S = \frac{d\,Q}{T} = 0.$

From (225) for $d\,U_1 = d\,U_2 = d\,W_2 = 0$, we have,

(267) $$Q_2 - Q_1 + W_1 = 0.$$

(268) From the first law. $d\,Q_2 - d\,Q_1 + d\,W_1 = 0.$

(269) From the second law. $d\,S_2 - d\,S_1 = \frac{d\,Q_2}{T_2} - \frac{d\,Q_1}{T_1} = 0.$

(270) Solving these equations. $d\,Q_1 = \frac{T_1}{T_1 - T_2}d\,W_1.$

$$d\,Q_2 = \frac{T_2}{T_1 - T_2}d\,W_1.$$

(271) $\frac{Q_1}{Q_2} = \frac{T_1}{T_2}. \quad \frac{Q_1}{Q_1 - Q_2} = \frac{T_1}{T_1 - T_2}. \quad \frac{Q_2}{Q_1 - Q_2} = \frac{T_2}{T_1 - T_2}.$

(272) $\frac{W_1}{Q_1} = \frac{(T_1 - T_2)}{T_1} = 1 - \frac{T_2}{T_1}.$ The efficiency of the engine.

(273) $\frac{W_1}{T_1} = Q_2\left(\frac{1}{T_2} - \frac{1}{T_1}\right). \quad \frac{W_1}{T_2} = Q_1\left(\frac{1}{T_2} - \frac{1}{T_1}\right).$

(274) Carnot's Function $= \frac{1}{T_1}$ in mechanical units for $T_1 > T_2$,

$= \frac{1}{A\,T_1}$ in heat units.

T_1 = the temperature of the source of heat energy.
T_2 = the temperature of the sink of the energy.
The energy runs down from the source to the sink.

For the irreversible process. $d\,S = \frac{d\,Q}{T} > 0.$

(275) From the first law. $d\,Q_2 - d\,Q_1 + d\,W_1 = 0.$

(276) From the second law. $d\,S_2 - d\,S_1 = \frac{d\,Q_2}{T_2} - \frac{d\,Q_1}{T_1} > 0.$

Solving these equations.

(277) $$d\,W_1 = (d\,Q_1 - d\,Q_2) < \left(d\,Q_1 - \frac{T_2}{T_1}d\,Q_1\right)$$

$$< d\,Q_1\left(1 - \frac{T_2}{T_1}\right) < d\,Q_1\frac{T_1 - T_2}{T_1}.$$

(278) For $dW_1 = 0,\ 0 < dQ_1\left(\frac{1}{T_2} - \frac{1}{T_1}\right) = \frac{dQ_2}{T_2} - \frac{dQ_1}{T_1}.$

(279) $dQ_1 > \frac{T_1}{T_1 - T_2}\, dW_1. \qquad dQ_2 > \frac{T_2}{T_1 - T_2}\, dW_1.$

Carnot's Cyclic Process

As an example of a reversible process we may describe the Carnot Cycle, in which a unit mass with the initial condition $(P_1.\ v_1.\ T_1)$ passes by an isothermal change to a second condition

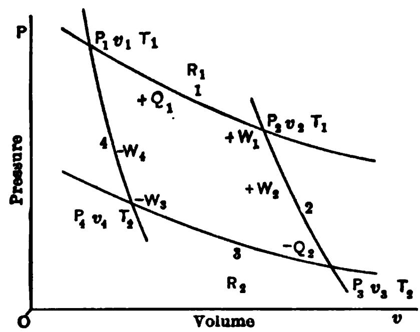


FIG. 3. Carnot's cycle

$(P_2.\ v_2.\ T_1)$, then by an adiabatic change to the third condition $(P_3.\ v_3.\ T_2)$, thence reversing from this extreme point by another isothermal change to $(P_4.\ v_4.\ T_2)$, and finally by another adiabatic change to the initial $(P_1.\ v_1.\ T_1)$. This is illustrated in Fig. 3.

T_1 = temperature of the source R_1.
T_2 = temperature of the sink R_2.
1 and 3 = Isothermal processes.
2 and 4 = Adiabatic processes.
1 and 2 = Work of expansion.
3 and 4 = Work of compression.
$+Q$ = Heat received.
$-Q$ = Heat expended.
$+W$ = Work of expansion.
$-W$ = Work of compression.

Summarizing by the first law.

	Isotherm.	*Adiab.*	*Isotherm.*	*Adiab.*

$$(280)\quad Q = Q_1 - Q_2 = W$$

$$= W_1 \qquad + W_2 \qquad - W_3 \qquad - W_4$$

$$(281)\quad = \int_{v_1 T_1}^{v_2 T_1} P\,d\,v \quad + \int_{v_2 T_1}^{v_3 T_2} P\,d\,v \quad - \int_{v_3 T_2}^{v_4 T_2} P\,d\,v \quad - \int_{v_4 T_2}^{v_1 T_1} P\,d\,v,$$

$$\cdot(282)\quad \text{Since } P = \frac{R\,T}{v} \quad = -\,Cv\,\frac{d\,T}{d\,v} \text{ by (228) for } d\,Q = 0.$$

$$= R\int_{v_1}^{v_2} \frac{T_1}{v} d\,v - \int_{T_1}^{T_2} Cv\; d\;T + R\int_{v_3}^{v_4} \frac{T_2}{v} d\,v - \int_{T_2}^{T_1} Cv\; d\;T,$$

$$(283)\quad \text{Since } R\log v + Cv\log T = \text{Const. by (228) for } d\,Q = 0.$$

$$= R\left(T_1 \log\frac{v_2}{v_1} + T_2 \log\frac{v_4}{v_3}\right),$$

$$(284)\quad = R\,(T_1 - T_2)\log\frac{v_2}{v_1} = R\,(T_1 - T_2)\log\frac{v_3}{v_4}.$$

It follows that $\log\frac{v_2}{v_1} = \log\frac{v_3}{v_4}$, or $\frac{v_2}{v_1} = \frac{v_3}{v_4}$.

$$(285)\quad (k-1)\log \rho_1 = \log T_1. \quad \text{By (180).}$$

$$(286)\quad +\,(k-1)\log v_1 + \log T_1 = (k-1)\log v_4 + \log T_2.$$

$$(287)\quad +\,(k-1)\log v_2 + \log T_1 = (k-1)\log v_3 + \log T_2.$$

From the Second Law by (271) and (180),

$$(288)\quad \frac{Q_2}{Q_1} = \frac{T_2}{T_1} = \left(\frac{v_1}{v_4}\right)^{k-1} = \left(\frac{v_2}{v_3}\right)^{k-1}.$$

$$Q_1 = R\,T_1 \log\frac{v_2}{v_1} = R\,T_1 \log\frac{v_3}{v_4}.$$

$$Q_2 = R\,T_2 \log\frac{v_2}{v_1} = R\,T_2 \log\frac{v_3}{v_4}.$$

Latent Heat

Cyclic Process for Vapors at Maximum Pressure

A second example of the reversible process is found in the cycle through which vapors pass in changing to liquids, by the latent heat which is required in effecting this transformation.

	Vapor	*Liquid*	*Solid*	*Total*
Mass	M_1	M_2	M_3	$M = M_1 + M_2 =$ Constant.
Specific Heat	C_1	C_2	C_3	
Latent Heat	...	r_1	r_2	
Volume	v_1	v_2	v_3	$v = v_1 + v_2$.

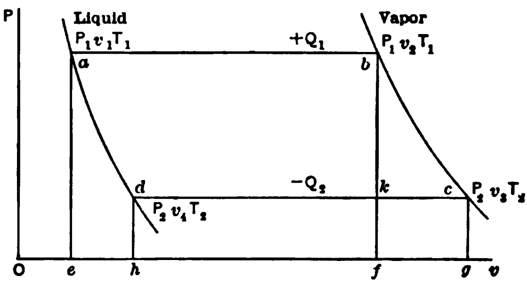


FIG. 4. Cyclic process for maximum vapor pressure

(289) Product. $M(v_1 + v_2) = M_1 v_1 + M_2 v_2 = M_1 v_1 + (M - M_1) v_2$.

(290) $$M v = M v_2 + M_1 (v_1 - v_2).$$

(291) $$d M v = M d v = (v_1 - v_2) d M_1.$$

This is the mass which evaporates in the expansion while $(v_3 - v_4) d M_2$ condenses during compression. Hence

(292) $$\frac{d M_1}{d v} = \frac{M}{v_1 - v_2}$$

The general equation of condition is,

(293) $$A P d v = r_2 \frac{d M_1}{d v} d v = r_2 \frac{M}{v_1 - v_2} d v \text{ in heat units.}$$

For $d v = 0$ we have by (209), $P = T \frac{d P}{d T}$. Hence,

(294) $$A P d v = A T \frac{d P}{d T} d v = r_2 \frac{M}{v_1 - v_2} d v, \text{ so that,}$$

(295) $$r_2 = A T_1 \frac{(v_1 - v_2)}{M} \frac{d P_1}{d T_1},$$ latent heat of vaporization of liquid to vapor $(v_1 - v_2)$ in heat units.

(296) $$r_3 = A T_2 \frac{(v_2 - v_3)}{M} \frac{d P_2}{d T_2},$$ latent heat of melting of solid to liquid $(v_2 - v_3)$ in heat units.

The Second Form of the Equations for Latent Heat

For *external equilibrium* where there is no exchange with the surrounding medium, the conditions are:

		Total	Vapor	Liquid	Solid
(297)	Masses.	M	$= M_1$	$+ M_2$	$+ M_3$.
(298)	Specific volumes.	v	$= v_1$	$+ v_2$	$+ v_3$.
(299)	Volume.	$M v$	$= M_1 v_1$	$+ M_2 v_2$	$+ M_3 v_3$.
(300)	Energy.	$M U$	$= M_1 U_1$	$+ M_2 U_2$	$+ M_3 U_3$.
(301)	Entropy.	$M S$	$= M_1 S_1$	$+ M_2 S_2$	$+ M_3 S_3$.

For *internal equilibrium* the general equation is,

(302) $A^1 M \delta S = A^1 \Sigma M_a \delta S_a + A^1 \Sigma S_a \delta M_a,$

Where a takes the values 1. 2. 3. in succession.

Since $U = Q - W$, and $T S = U + W = U + P d v = Q$, we have by differentiation and substitution,

$$(303) \qquad \delta S = \frac{\delta U}{T} + \frac{P \delta v}{T},$$

$$(304) \quad A^1 M \delta S = \Sigma \frac{M_a \delta U_a}{T_a} + \Sigma \frac{M_a P_a \delta v_a}{T_a} + A^1 S_a \delta M_a = 0.$$

The three independent conditions for interpreting this equation are,

(305) For the masses, $\Sigma \delta M_a = 0.$

(306) For the volumes, $\Sigma M_a \delta v_a + \Sigma v_a \delta M_a = 0.$

(307) For the energies, $\Sigma M_a \delta U_a + \Sigma U_a \delta M_a = 0.$

After eliminating from the equations $\delta M_2 \,.\, \delta v_2 \,.\, \delta U_2$.

$$(308) \quad A^1 \delta S = \left(\frac{1}{T_1} - \frac{1}{T_2}\right) M_1 \delta U_1 - \left(\frac{1}{T_2} - \frac{1}{T_3}\right) M_3 \delta U_3$$
$$+ \left(\frac{P_1}{T_1} - \frac{P_2}{T_2}\right) M_1 \delta v_1 - \left(\frac{P_2}{T_2} - \frac{P_3}{T_3}\right) M_3 \delta v_3$$
$$+ \left[A^1(S_1 - S_2) - \frac{U_1 - U_2}{T^1} - \frac{P_1 (v_1 - v_2)}{T_1}\right] \delta M_1$$
$$- \left[A^1 (S_2 - S_3) - \frac{U_2 - U_3}{T_2} - \frac{P_2 (v_2 - v_3)}{T_2}\right] \delta M_3 = 0.$$

The conditions of equilibrium for the maximum entropy are uniform temperature, $T = T_1 = T_2 = T_3$, and uniform pressure, $P_1 = P_2 = P_3$.

Hence we have by selection,

(309) $r_2 = T_1 (S_1 - S_2) = A (U_1 - U_2) + P_1 (v_1 - v_2)$ for vapor and liquid.

(310) $r_3 = T_2 (S_2 - S_3) = A (U_2 - U_3) + P_2 (v_2 - v_3)$ for liquid and solid.

Specific Heats

A third example of a reversible cyclic process is given in the specific heats, C_1 for vapor, C_2 for fluid. From the first law by (234), we have,

(234) $$A \frac{dP}{dT} = \frac{d.AP}{dT} - \frac{d.Cv}{dv}.$$

Develop these terms successively by substitutions.

Differentiate the last form of (293), dividing by dv,

(311) $$(1) \frac{d.AP}{dT} = \frac{M}{v_1 - v_2} \cdot \frac{dr_2}{dT} - \frac{r_2 M d(v_1 - v_2)}{(v_1 - v_2)^2 dT} = \left[\frac{dr_2}{dT} - \frac{r_2 d(v_1 - v_2)}{(v_1 - v_2) dT}\right] \frac{M}{v_1 - v_2}$$

The specific heat of the mixture is, by (136),

(312) $$Cv = C_2 (M - M_1) + C_1 M_1 - r_2 \frac{dM}{dT}.$$ Hence, by (289),

(313) $$Cv = C_2 (M - M_1) + C_1 M_1 - \frac{r_2 d[M_1 v_1 + (M - M_1) v_2]}{(v_1 + v_2) dT}.$$

(314) $$(2) \frac{dCv}{dv} = -\frac{C_2 dM_1}{dv} + \frac{C_1 dM_1}{dv} - \frac{r_2}{(v_1 + v_2)} \left[\frac{dv_1 - dv_2}{dT}\right] \frac{dM_1}{dv}.$$ By (292),

(315) $$\frac{dCv}{dv} = \left[-C_2 + C_1 - \frac{r_2}{(v_1 + v_2)} \frac{d(v_1 - v_2)}{dT}\right] \frac{M}{(v_1 - v_2)}.$$

Subtract (1) − (2).

(316) $$A \frac{dP}{dT} = \left(\frac{dr_2}{dT} + C_2 - C_1\right) \frac{M}{v_1 - v_2}.$$

(317) $$\frac{d r_2}{d T} + C_2 - C_1 = A \frac{(v_1 - v_2)}{M} \frac{d P}{d T} \text{ in heat units.}$$

The Specific Heats in Terms of the Latent Heats

It can be proved by differentiation of the first forms of (309) and (310) and the necessary substitutions that,

(318) $$(Cp)_1 - (Cp)_2 = \frac{d r_2}{d T} - \frac{r_2}{T} + \frac{r_2}{v_1 - v_2}\left[\left(\frac{\partial v_1}{\partial T}\right)_P - \left(\frac{\partial v_2}{\partial T}\right)_P\right],$$
vapor-liquid.

(319) $$(Cp)_2 - (Cp)_3 = \frac{d r_3}{d T} - \frac{r_3}{T} + \frac{r_3}{v_2 - v_3}\left[\left(\frac{\partial v_2}{\partial T}\right)_P - \left(\frac{\partial v_3}{\partial T}\right)_P\right],$$
liquid-solid.

Compare Planck's "Thermodynamik."

Examples of the Thermodynamic Data

1. *Carnot's Cycle*. In Fig. 3 the area enclosed between the isotherms (1.3) and the adiabats (2.4) represents the work done, W, in the cyclic process, and the figure is called the indicator diagram. This is used in studying the efficiency of engines, whether the process is natural or mechanical, and there is a very large literature on the subject. No applications have as yet been made in the atmosphere, for two reasons, the first because of the difficulty of tracing out the history of a given mass, and the second because the values of Cv in the formula are not constant, and the true values of it have not heretofore been computed. However, it is possible to take a standard mass, as one kilogram of air at an initial point, and trace out the conditions through which it must have passed in rising from the surface through changing $P.\ v.\ T.$ till it arrives at the surface again in the original state, even though the path during its circulation may not be known. This work will be reserved for further studies.

2. *Cyclic Process for Vapors*. In Fig. 4, we have,

(320) $(v_2 - v_1)\, d M_1$ the mass which evaporates in expansion.

(321) $(v_3 - v_4)\, d M_2$ the mass which condenses in compression.

(322) $b\,k = \frac{dP}{dT} d\,T$ the increase in the pressure.

(323) $W = (v_1 - v_2) \frac{dP}{dT} . d\,M_1 d\,T$ the work done in the area a, b, c, d.

(324) $Q_1 = r_2\, d\,M_1 = C_2\, d\,M_1\, d\,T$ the heat received in expansion.

(325) $Q_2 = \left(r_2 - \frac{d\,r_2}{d\,T}\right) d\,M_1.$

(326) $Q_2 = C_1\, d\,M_1\, d\,T - \frac{d\,r_2}{d\,T} d\,M_1\; d\,T$ the heat expended in compression.

(327) $Q = Q_1 - Q_2 = \left(\frac{d\,r_2}{d\,T} + C_2 - C_1\right) d\,M_1\, d\,T.$

Since $A = \frac{Q}{W}$, this becomes as in (317),

(328) $\frac{d\,r_2}{d\,T} + C_2 - C_1 = A\;(v_1 - v_2)\;\frac{d\,P}{d\,T}$ in heat units.

Values of the Latent Heat and Specific Heats

(329) $r_2 = 606.5 - 0.708\,t$ for water to aqueous vapor.

$r_3 = 80.066$ for ice to water.

$Cp_1 = 0.4810 \qquad Cp_2 = 1.0000 \qquad Cp_3 = 0.5020.$

Example 3. *Water to Vapor at* 100° *C., by Vaporization*

$T_1 = 273 + 100 = 373.$

$v_1 = 1658$ the volume of 1 gram of aqueous vapor at 100° C.

$v_2 = 1$ the volume of 1 gram of water at 0°.

$v_1 - v_2 = 1658 - 1 = 1657.$

$\frac{d\,P_1}{d\,T} = 27.2$ millimeters of mercury.

$\frac{A}{g_0} = 2.3894 \times 10^{-8} = \frac{1}{41851000}$ (log = 2.37829 − 10)

$\frac{A}{g_0} \times \frac{1013235}{760} = 3.1856 \times 10^{-5}$ (log = 5.50319 − 10) reduction from work units and mm. to C. G. S. units heat.

(295) Latent Heat. $r_2 = \frac{A}{g_0} T_1 (v_1 - v_2) \frac{d P_1}{d T} = 373 \times 1657 \times$
$27.2 \times 3.1856 \times 10^{-5} = 535.5.$

Example 4. *Ice to Water at* 0° *by Melting*

$T_2 = 273 + 0 = 273.$
$v_2 = 1.00$ volume of 1 gram of water at 0°.
$v_3 = 1.09$ volume of 1 gram of ice at 0°.
$v_2 - v_3 = - 0.09.$
$- \frac{d P_2}{d T} = 134.6$ atmospheres.

(296) Latent Heat. $r_3 = \frac{A}{g_0} . T_2 (v_2 - v_3) \frac{d P_2}{d T} = 273 \times (- 0.09)$
$(-134.6) \times 3.1856 \times 10^{-5} \times 760 = 80.066.$

Example 5. *Aqueous Vapor in Contact with Water at* 100°.

$\frac{d r_2}{d T} = - 0.708 \qquad \left(\frac{\partial v_1}{\partial T}\right)_P = 4.931 \qquad \left(\frac{\partial v_2}{\partial T}\right)_P = 0.001$

$Cp_2 = 1.0300 \qquad P = 760 \text{ mm.} \qquad T = 373$

(318) $Cp_1 - Cp_2 = \frac{dr_2}{T} - \frac{r_2}{T} + \frac{r_2}{v_1 - v_2} \left[\left(\frac{\partial v_1}{\partial T}\right)_P - \left(\frac{\partial v_2}{\partial T}\right)_P \right]$
$= - 0.708 - \frac{535.5}{373} + \frac{535.5}{1657} [4.931 -$
$0.001] = - 0.5504.$

Specific Heat. $Cp_1 = 1.0300 - 0.5490 = 0.4810.$

Example 6. *Water in Contact with Ice at* 0°

$\frac{d r_3}{d T} = 0.6400 \qquad \left(\frac{\partial v_2}{\partial T}\right)_P = - 0.00006 \qquad \left(\frac{\partial v_3}{\partial T}\right)_P = + 0.00011.$

$Cp_2 = 1.0000$ water at 0°.

(319) $Cp_2 - Cp_3 = \frac{d r_3}{d T} - \frac{r_3}{T} + \frac{r_3}{v_2 - v_3} \left[\left(\frac{\partial v_2}{\partial T}\right)_P - \left(\frac{\partial v_3}{\partial T}\right)_P \right]$
$= 0.6400 - \frac{80}{273} + \frac{80}{(-0.09)} [- 0.00006 -$
$0.00011]$
$= 0.6400 - 0.2930 + 0.1510 = 0.4980.$

Specific Heat. $Cp_3 = 1.0000 - 0.4980 = 0.5020.$

Example 7. *Pressure of Vapor in Contact with Water and Ice*

$P_{12} = 4.57$ vapor-water. $v_1 = 205000.$

$P_{13} = 760$ vapor-ice. $v_2 = 1.00$

$P_{23} =$ water-ice. $v_3 = 1.09$

$T = 0.0074°$ C., the fundamental temperature.

(295) Vapor on water. $$\frac{d P_{12}}{d T} = \frac{r_{12}}{T (v_1 - v_2)} \frac{41851000}{1} \frac{760}{1013235}.$$

$$= \frac{606.5}{273} \cdot \frac{3.1391 \times 10^4}{204999} = 0.3402 \text{ mm.}$$

(295) Vapor on ice. $$\frac{d P_{13}}{d T} = \frac{g_0 r_{13}}{A T (v_1 - v_3)} =$$

$$\frac{(606.5 + 80)}{273} \cdot \frac{3.1391 \times 10^4}{204999} = 0.3851 \text{ mm.}$$

(296) Water on ice. $$\frac{d P_{23}}{d T} = \frac{g_0 r_{23}}{A T (v_2 - v_3)} =$$

$$\frac{80.066}{273} \cdot \frac{3.1391 \times 10^4}{(-0.09)} = -102294. \text{ mm.}$$

Compare Planck's "Thermodynamik."

Application of the Thermodynamic Formulas to the Non-Adiabatic Atmosphere

The foregoing formulas would apply to an adiabatic atmosphere, using the constants of Table 3, wherein *Cp*, *Cv*, *R* are constants, but they do not apply to the existing non-adiabatic atmosphere, because it is not an ideal gas, rather a mixture of gases which are undergoing rapid changes of condition through variations in the heat contents by insolation and radiation. They can, however, be adapted to the earth's atmosphere by suitable modifications, which depend upon the formulas developed under static meteorology. The following summary is sufficient for working purposes.

Entropy

(330) $$dS = \frac{dQ}{T} = Cp\frac{dT}{T} - R\frac{dP}{P} = Cp\frac{dT}{T} - \frac{v}{T}dP.$$

(331) $$S_1 - S_0 = \frac{Q_1 - Q_0}{T_{10}} = n_1 Cp_a \frac{T_1 - T_0}{T_{10}} - \frac{1}{M} R_{10} \log \frac{P_1}{P_0} =$$
$$n_1 Cp_a \frac{1}{M} \log \frac{T_1}{T_0} - \frac{1}{M} R_{10} \log \frac{P_1}{P_0}.$$

Work Against External Forces

(332) $$dW = P\,dv = R\,dT - \frac{dP}{\rho} = Cp\frac{k-1}{k}dT - \frac{dP}{\rho}.$$

(333) $$W_1 - W_0 = P_{10}(v_1 - v_0) = R_{10}(T_a - T_0) - \frac{P_1 - P_0}{\rho_{10}}$$
$$= Cp_{10}\frac{k-1}{k}(T_a - T_0) - \frac{P_1 - P_0}{\rho_{10}}$$
$$= \left(\frac{k-1}{k} - 1\right)\frac{P_1 - P_0}{\rho_{10}}.$$

Inner Energy

(334) $$dU = Cv\,dT = (Cp - R)\,dT = Cp\,dT - (P\,dv + v\,dP)$$
$$= Cp\,dT - \frac{k-1}{k}\frac{dP}{\rho}.$$

(335) $$U_1 - U_0 = Cv_{10}(T_a - T_0) = (Cp_a - R_{10})(T_a - T_0) =$$
$$Cp_a(T_a - T_0) - Cp_{10}\frac{k-1}{k}(T_a - T_0)$$
$$= (Q_1 - Q_0) - (W_1 - W_0) = Cp_a(T_a - T_0) -$$
$$\frac{k-1}{k}\frac{P_1 - P_0}{\rho_{10}}.$$

Heat Energy

(336) $$dQ = Cv\,dT + RT\frac{dv}{v} = Cv\,dT + Cp\frac{k-1}{k}dT -$$
$$\frac{dP}{\rho} = Cp\,dT - \frac{dP}{\rho}$$
$$= T\,dS \qquad = Cp\left(\frac{k-1}{k} + 1\right)dT -$$
$$\left(\frac{k-1}{k} + 1\right)\frac{dP}{\rho}.$$

(337) $Q_1 - Q_0 = T_{10}(S_1 - S_0) = (Cp_a - Cp_{10})(T_a - T_0) = Cv_{10}(T_a - T_0) + P_{10}(v_1 - v_0).$
$= (U_1 - U_0) + (W_1 - W_0) = Cp_a (T_a - T_0) - \dfrac{P_1 - P_0}{\rho_{10}}.$

Radiation Function

(338) $dK = \dfrac{dU}{dv} = \dfrac{dQ}{dv} - P = \dfrac{T\,dS}{dv} - P = -P\dfrac{R_a}{R}.$

(339) $K_{10} = \dfrac{U_1 - U_0}{v_1 - v_0} = \dfrac{Q_1 - Q_0}{v_1 - v_0} - P_{10} = \dfrac{T_{10}(S_1 - S_0)}{v_1 - v_0} - P_{10}$
$= -P_{10}\dfrac{R_a}{R_{10}}.$

(340) $\dfrac{K_1}{K_0} = \left(\dfrac{T_1}{T_0}\right)^A.$

(341) $\log K_1 - \log K_0 = A(\log T_1 - \log T_0).$

(342) $A = \dfrac{\log K_1 - \log K_0}{\log T_1 - \log T_0}.$

The Radiation Coefficients and Exponents

(343) $K_{10} = C\,T_{10}^{A}.$ $\log K_{10} = \log C + A \log T_{10}.$

(344) $K_{10} = c\,T_{10}^{a}.$ $\log K_{10} = \log c + a \log T_{10}.$

(345) $\log c = \log C_0 + (A - 4)\log B.$
$C_0 = 9.12 \times 10^{-5}.$ $B = 1.66 \times 10^{-2}.$

(346) $\log c = -5.906 - (2.220)(A - 4).$

(347) $c = C_0 B^{A-4} = 9.12 \times 10^{-5}(1.66 \times 10^{-2})^{A-4}.$

These formulas will be fully explained and illustrated in the examples that follow (pages 84–85).

Working Equations

(331) $S_1 - S_0 = \dfrac{Q_1 - Q_0}{T_{10}}.$

(333) $W_1 - W_0 = R'_{10}(T_a - T_0) - \dfrac{P_1 - P_0}{\rho_{10}}.$

(335) $U_1 - U_0 = (Q_1 - Q_0) - (W_1 - W_0).$

(339) $$K_{10} = \frac{U_1 - U_0}{v_1 - v_0}.$$

(340) $$\frac{K_1}{K_0} = \left(\frac{T_1}{T_0}\right)^A.$$ (342) $$A = \frac{\log K_1 - \log K_0}{\log T_1 - \log T_0}.$$

In order to illustrate the formulas of computation (331) to (342) the data of Table 17 are continued in Table 19. It must be especially noted that $-(Cp_a - Cp_{10})(T_a - T_0)$, which by (198) includes the kinetic energy of circulation and the kinetic energy of radiation $-\frac{1}{2}(q_1^2 - q_0^2) - (Q_1 - Q_0)$, is not carried forward, but only $(Q_1 - Q_0)$, the energy of radiation. If the former were taken for the computations beyond this point the circulation would be treated as true radiation, which is improper. The sign $-(Q_1 - Q_0)$ in Table 17 is changed to $+(Q_1 - Q_0)$ in Table 19. In computing the mean entropy from one level to another, the mean temperature $T_{10} = \frac{1}{2}(T_1 + T_0)$ is taken from Table 14 in successive pairs. The entropy generally increases with the height, and always does so unless there is an inversion of temperature, or an excess of wind variation in velocity between the levels, such as occurred in this case between 500 and 1,000 meters.

In applying (333) for computing the work $(W_1 - W_0)$ we must now compute R'_{10} corresponding with $(Q_1 - Q_0)$, which differs from R_{10} taken by pairs in Table 14, since R_{10} implies the circulation as well as the radiation. The formulas now become

(348) $(Cp_a - Cp'_{10})(T_a - T_0) = (Q_1 - Q_0).$

(349) $$Cp'_{10} = Cp_a - \frac{Q_1 - Q_0}{T_a - T_0} = 993.58 - \frac{Q_1 - Q_0}{T_a - T_0}.$$

(350) $$R'_{10} = Cp'_{10}\frac{k-1}{k}.$$

TABLE 19. EXAMPLE OF THE COMPUTATION OF THE THERMODYNAMIC TERMS $(S_1 - S_0)$, $(W_1 - W_0)$, $(U_1 - U_0)$, K_{10}, A.

Lindenberg, April 27, 1909

z	116	500	1000	1500	2000	2500	3000	4000	5000
+ $(Q_1 - Q_0)$	......	−54.2	+ 5.4	−23.6	−91.4	−155.2	−197.8	−442.0	−620.0
log $(Q_1 - Q_0)$	......	− 1.73400	0.73239	− 1.37291	− 1.96095	− 2.19089	2.29623	− 2.64542	− 2.79239
T_{10}	......	292.25	288.30	284.90	282.15	279.85	277.20	272.40	266.00
log T_{10}	......	2.46576	2.45984	2.45469	2.45048	2.44693	2.44279	2.43521	2.42488
log $(S_1 - S_0)$	......	− 9.26824	8.27255	− 8.91822	9.51047	− 9.74396	− 9.85344	− 0.21021	− 0.36751
$S_1 - S_0$	......	− 0.185	+ 0.018	− 0.083	− 0.324	− 0.555	− 0.714	− 1.623	− 2.331
log $(T_a - T_0)$	......	− 0.57862	− 0.69326	− 0.69326	−0.69326	− 0.69326	− 0.69326	− 0.99429	− 0.99429
log $(Cp_a - Cp'_{10})$	......	1.15538	− 0.03913	0.67965	1.26769	1.49763	1.60297	1.65113	1.79810
$(Cp_a - Cp'_{10})$	......	+14.30	− 1.09	+ 4.78	+18.52	+31.45	+ 40.08	+44.78	+62.82
Cp'_{10}	......	979.28	994.67	988.80	975.06	962.13	953.50	948.80	930.76
log Cp'_{10}	......	2.99091	2.99768	2.99511	2.98903	2.98324	2.97932	2.97717	2.96884
log R'_{10}	......	2.45164	2.45841	2.45584	2.44976	2.44397	2.44005	2.43790	2.42957
R'_{10}	......	282.91	287.35	285.65	281.68	277.95	275.46	274.09	268.89
log $R'_{10}(T_a - T_0)$	......	− 3.03026	− 3.15167	− 3.14910	− 3.14302	− 3.13723	− 3.13331	− 3.43219	− 3.42386
$R'_{10}(T_a - T_0)$	......	−1072.2	−1418.0	−1409.6	−1390.0	−1371.6	−1359.3	−2705.1	−2653.7
$\frac{P_1 - P_0}{\rho_{10}}$	......	+3768.5	+4904.2	+4867.4	+4834.2	+4792.2	+4750.6	+9418.5	+9289.0
$W_1 - W_0$	......	+2696.3	+3486.2	+3457.8	+3444.2	+3420.6	+3391.3	+6713.4	+6635.3
$U_1 - U_0$	......	−2750.5	−3480.8	−3481.4	−3535.6	−3575.8	−3589.1	−7155.4	−7255.3

log $(U_1 - U_0)$	......	− 3.43941	−3.54168	− 3.54175	− 3.54844	− 3.55338	− 3.55498	−3.85463	−3.86066
log ρ	0.07088	0.05702	0.03870	0.02021	0.00151	9.98266	9.96363	9.92487	9.88522
log v	9.92912	9.94298	9.96130	9.97979	9.99849	0.01734	0.03637	0.07513	0.11478
v	0.8494	0.8770	0.9147	0.9545	0.9965	1.0407	1.0874	1.1889	1.3025
$v_1 - v_0$	......	+ .0276	+ .0377	+ .0398	+ .0420	+ .0442	+ .0467	+ .1015	+ .1136
log $(v_1 - v_0)$	......	8.44091	8.57634	8.59988	8.62325	8.64542	8.66932	9.00647	9.05538
log K_{10}	......	− 4.99850	− 4.96534	− 4.94187	− 4.92519	− 4.90796	− 4.88566	− 4.84816	− 4.80528
K_{10}	......	−99655	−92330	−87472	−84176	−80902	−76853	−70495	−63867
log K_1 − log K_0	......	....	− .03316	− .02347	− .01668	− .01723	− .02230	− .03750	− .04288
log (log K_1 − log K_0)	......	....	− 8.52061	− 8.37051	− 8.22220	− 8.23629	− 8.34830	− 8.57403	− 8.63225
log (log T_1 − log T_0	......	....	− 7.73480	− 7.68753	− 7.54900	− 7.55267	− 7.67210	− 8.02243	− 8.00518
log A	......	....	0.78581	0.68298	0.67320	0.68362	0.67620	0.55160	0.62707
A	......	....	6.11	4.82	4.71	4.83	4.74	3.56	4.24

COMPARISON OF R_{10} WITH R'_1

Height z	116	500	1000	1500	2000	2500	3000	4000	5000
R_{10}	...	287.28	286.87	285.34	283.14	280.50	278.21	275.59	272.06
R'_{10}	...	282.91	287.35	285.65	281.68	277.95	275.46	274.09	268.89

R'_{10} is generally smaller than R_{10} in these observations.

The term $\dfrac{P_1 - P_0}{\rho_{10}}$ is taken directly from Table 17, and $W_1 - W_0$ is easily computed. Then $U_1 - U_0$ follows from (335).

In computing the radiation, $K_{10} = \dfrac{U_1 - U_0}{v_1 - v_0}$, the values of v are the reciprocal of the density ρ in Table 14. Had ρ been computed by formula (175), which takes R constant, instead of by (176), with R variable, it is seen at once how erroneous would have been the derived radiations, because the values of K_{10} depend upon the small differences $(v_1 - v_0)$ in succession. These radiations are mean values for the strata concerned. It is important to study the relation of the radiation to the temperature, and to compare the exponents of formula (340) with the exponent of a full radiation in the Stephan Law, which is 4. This subject is complex in the earth's atmosphere as will be indicated. The problem is as follows: The values of K in relation to T by (340) are in the form of ratios, whereas in the Stephan Law (344) they stand related through a coefficient. If the constituent of the ratio is in the form (343), it is quite certain that the coefficients $(C \,.\, c)$ are not equal, nor are the exponents $(A \;.\; a)$. We proceed to develop the relations between C and c, A and a. The equation (343) gives three terms, K_{10}, A, T_{10}, from which to compute C, and it is necessary to indicate what are the relative values of $\log C$ and A. With the data of Table 19 in the first section of Table 20, compute $A \log T_{10}$ and subtract this from $\log K_{10}$ to obtain $\log C$. The negative sign before the logarithm affects only the characteristic. Thus, logarithm $-$ 11.944 gives the number 8.79×10^{-11}. In this way the values of log C and A were computed for twelve balloon ascensions, of which two examples are given in Table 21. It is readily seen that log C is negative, as the temperature T decreases with the height z, and positive in regions of inversion of temperature. The magnitude of log C depends upon the ratio $\left(\frac{T_1}{T_0}\right)$. If the temperature changes slowly with the height the

TABLE 20

Computation of Log C and (a . Log c)

Section I. $\log C = \log K_{10} - A \log T_{10}$ (343)

Lindenburg, April 27, 1909

z	116	500	1000	1500	2000	2500	3000	4000	5000
T_{10}	...	292.25	288.30	284.90	282.15	279.85	277.20	272.40	266.00
log T_{10}	...	......	2.45984	2.45469	2.45048	2.44693	2.44279	2.43521	2.42488
log.log T_{10}	...	......	0.39091	0.39000	0.38925	0.38862	0.38789	0.38654	0.38469
log A	...	......	0.78581	0.68298	0.67320	0.68362	0.67620	0.55160	0.62707
A	...	......	6.11	4.82	4.71	4.83	4.74	3.56	4.24
log (A log T_{10})	...	......	1.17672	1.07298	1.06245	1.07224	1.06409	0.93814	1.01176
A log T_{10}	...	......	15.021	11.830	11.547	11.810	11.590	8.672	10.275
log K_{10}	...	......	4.96534	4.94187	4.92519	4.90796	4.88566	4.84816	4.80528
K_{10}	...	......	92330	87472	84176	80902	76853	70495	63867
log C	...	......	−11.944	−7.112	−7.378	−7.098	−7.246	−4.176	−6.530
C	...	......	8.79×10^{-11}	1.29×10^{-7}	2.39×10^{-7}	1.25×10^{-7}	1.76×10^{-7}	1.50×10^{-4}	3.39×10^{-6}

Section II. $\log c = \log K_{10} - a \log T_{10}$ (344)

Assumed a_1	...	......	3.82	3.82	3.82	3.82	3.82	3.81	3.81
log a_1	...	......	0.5821	0.5821	0.5821	0.5821	0.5821	0.5809	0.5809
log.log T_{10}	...	......	0.3909	0.3900	0.3893	0.3886	0.3879	0.3865	0.3847
log a_1 log T_{10}	...	......	0.9730	0.9721	0.9714	0.9707	0.9700	0.9674	0.9656
a_1 log T_{10}	...	......	9.397	9.378	9.363	9.348	9.333	9.277	9.238
log K_{10}	...	......	4.965	4.942	4.925	4.908	4.886	4.848	4.805
log c	...	......	−5.568	−5.564	−5.562	−5.560	−5.508	−5.571	−5.567
a_0	...	......	3.824	3.822	3.821	3.820	3.794	3.825	3.823
Second Assumed a_2	...	......	....	....	....	....	3.807	3.817	3.817
log a_2	...	......	....	....	....	....	0.5806	0.5817	0.5817
log.log T_{10}	...	......	....	....	....	....	0.3879	0.3865	0.3847
log . a_2 log T_{10}	...	......	....	....	....	....	0.9685	0.9682	0.9664
a_2 log T_{10}	...	......	....	....	....	....	9.300	9.294	9.256
log K_{10}	...	......	....	....	....	....	4.836	4.848	4.805
log c	...	......	....	....	....	....	−5.536	−5.554	−5.549
a	...	......	....	....	....	....	3.809	3.817	3.815

Working Formulas

(343) $K_{10} = C T_{10}^{A}$

$\log K_{10} = \log C + A \log T_{10}$

$\log C = \log K_{10} - A \log T_{10}$

(344) $K_{10} = c T_{10}^{a}$

$\log K_{10} = \log c + a \log T_{10}$

$\log c = \log K_{10} - a \log T_{10}.$

Assume trial a_1 and adjust by Table 22. The final pair log c and a should fall on the same line.

TABLE 21

EXAMPLES OF LOG C . A AND LOG c . a THE COEFFICIENTS AND EXPONENTS IN THE RADIATION FORMULAS (343), (344)

z	Lindenburg, April 27, 1909				Lindenburg, May 5, 1909			
	log C	A	log c	a	log C	A	log c	a
17000	....	...	......	.....	....	......	......	....
16000	....	...	......	.....	+28.034	−10.34	−5.435	3.764
15000	....	...	......	.....	+52.871	−21.11	−5.461	3.775
14000	....	...	......	.....	+16.828	− 5.78	−5.489	3.788
13000	− 4.356	3.45	−5.531	3.807	+10.437	− 3.41	−5.537	3.810
12000	− 7.140	4.86	−5.563	3.822	−27.533	13.29	−5.563	3.821
11000	− 5.962	3.64	−5.571	3.825	− 6.181	4.40	−5.548	3.814
10000	− 1.410	3.44	−5.560	3.820	− 3.288	3.09	−5.558	3.819
9000	− 2.114	2.73	−5.549	3.815	− 4.021	3.62	−5.558	3.819
8000	− 3.211	3.11	−5.539	3.810	− 6.506	4.26	−5.560	3.820
7000	− 7.178	4.79	−5.538	3.810	− 5.249	3.95	−5.557	3.818
6000	− 9.719	5.40	−5.548	3.814	− 5.401	3.89	−5.560	3.820
5000	− 6.530	4.24	−5.549	3.815	−11.132	6.48	−5.547	3.814
4000	− 4.176	3.56	−5.554	3.817	− 4.050	3.63	−5.560	3.820
3000	− 7.246	4.74	−5.536	3.809	− 8.115	5.25	−5.567	3.823
2500	− 7.098	4.83	−5.560	3.820	−18.387	9.23	−5.577	3.828
2000	− 7.378	4.71	−5.562	3.821	−20.982	9.81	−5.590	3.833
1500	− 7.112	4.82	−5.564	3.822	−14.180	7.64	−5.601	3.838
1000	−11.944	6.11	−5.568	3.824	+24,708	−8.29	−5.593	3.834
500	....	...	......	.....	....	......	......	....

ratio is small. At the same time the ratio $\left(\frac{K_1}{K_6}\right)$, which does not have a coefficient or exponent, registers a change in the atmosphere which is closely connected with the variation of the pressure P. Hence there are large changes in $\log C$ and A which are opposite in sign, but both increasing or diminishing together. Under nearly normal conditions it is seen that A is approximately 4.00, which would be the value for a full radiating black body. The entire series of values A, $\log C$, were collected in groups, and the mean values when plotted fall on a straight line, of which the equation was found to be,

(347) $C = C_0 B^{A-4}$.
(345) $\log C = \log C_0 + (A - 4) \log B$.
(346) $\log C = -5.960 + (A - 4)(-2.220)$.
(347) $C = 9.12 \times 10^{-5} (1.66 \times 10^{-2})^{A-4}$.

The development of a portion of this formula from $A = 4.00$ to $A = 3.50$ is given in Table 22. $A = 4.00$ corresponds with a full black radiator and $A = 3.50$ corresponds with the theoretical value for the atmospheric air. The mean value was found to be $A = 3.82$ near the surface. The notation ($\log c$. a) indicates the values in the constituent formula, while ($\log C$. A) are used for the ratio formula.

TABLE 22

EVALUATION OF THE FORMULA (346)

a	$\log c$	a	$\log c$	a	$\log c$	a	$\log c$
4.00	−5.960	3.85	−5.627	3.70	−5.294	3.55	−6.961
3.99	−5.938	3.84	−5.605	3.69	−5.272	3.54	−6.939
3.98	−5.916	3.83	−5.583	3.68	−5.250	3.53	−6.917
3.97	−5.893	3.82	−5.560	3.67	−5.227	3.52	−6.894
3.96	−5.871	3.81	−5.538	3.66	−5.205	3.51	−6.872
3.95	−5.849	3.80	−5.516	3.65	−5.183	3.50	−6.850
3.94	−5.827	3.79	−5.494	3.64	−5.161	......	......
3.93	−5.805	3.78	−5.472	3.63	−5.139	......	......
3.92	−5.782	3.77	−5.449	3.62	−5.116	......	......
3.91	−5.760	3.76	−5.427	3.61	−5.094	......	......
3.90	−5.738	3.75	−5.405	3.60	−5.072	......	......
3.89	−5.716	3.74	−5.383	3.59	−5.050	......	......
3.88	−5.694	3.73	−5.361	3.58	−5.028	......	......
3.87	−5.671	3.72	−5.338	3.57	−5.005	......	......
3.86	−5.649	3.71	−5.316	3.56	−6.983	......	......

The negative sign applies only to the characteristic

$$\log C_0 = -5.960 \quad C_0 = 9.12 \times 10^{-5}$$
$$\log B = -2.220 \quad B = 1.66 \times 10^{-2}.$$

By means of Table 22 we proceed in Section II of Table 20 to compute $\log c$ and a from $\log C$ and A. Assume an approximate value a_1, as 3.82, and compute $a_1 \log T_{10}$ from the value in Section I. Subtract from $\log K_{10}$ for $\log c$, and in Table 22 interpolate that value a_0, which is the pair value of $\log c$. If this value a_0 agrees with the assumed a_1 the check is complete. If, on the other hand, these values of $\log c$ and a do not quite agree, as in the examples under $z = 3000, 4000, 5000$, take the mean value between the assumed a_1 and computed a_0, and proceed again with a_2 to compute $\log c$ and a. The second trial is usually successful if a_1 has been chosen with some practice. The corresponding values of ($\log c$. a) are found in the examples of Table 21, and it is seen that the irregularities of ($\log C$. A) have disappeared. Log c and a usually decrease slowly with the elevation and with the increase of latitude from the equator.

These results check by $\log c + a \log T_{10} = \log K_{10}$.

Application of the Thermodynamic Formulas to Various Meteorological Problems

It has seemed necessary to give an extended example of the method of computing the thermodynamic values in the non-adiabatic atmosphere, on account of the complexity of the computations, and because of the numerous valuable results dependent upon them. In Bulletin No. 3 of the Argentine Meteorological Office, 1913, will be found the results for many types of data in considerable detail. We can here summarize them briefly, depending upon diagrams to bring out the general ideas, in particular respecting the isothermal region, the diurnal convection, the circulation in cyclones and anti-cyclones, and the general circulation of the atmosphere.

The Isothermal Region

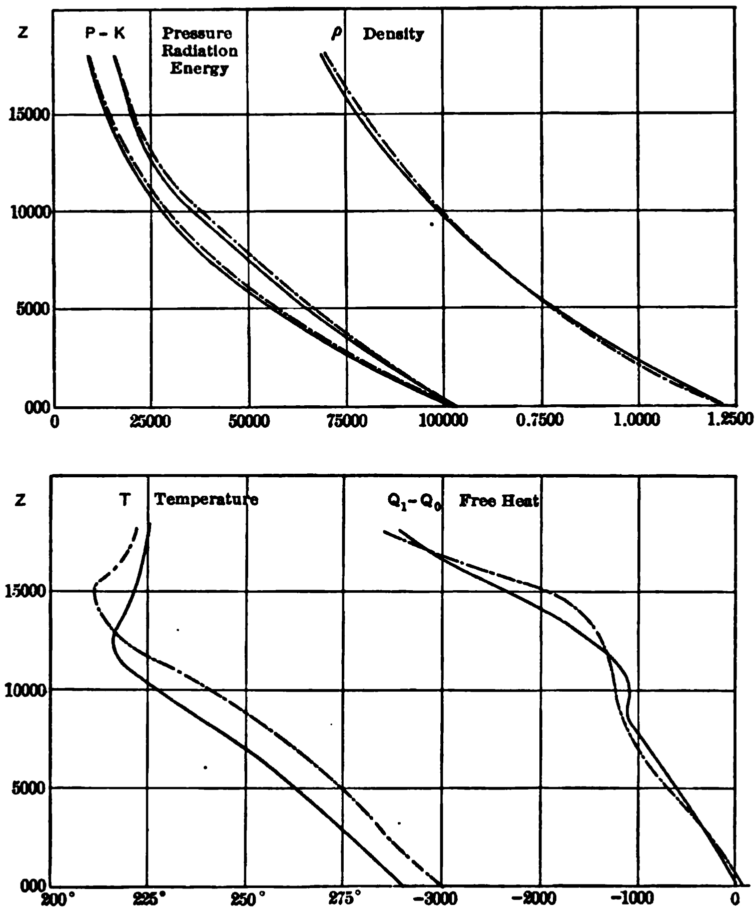
It has been found by balloon ascensions to great elevations, up to 20,000 meters or more, that the temperature of the atmosphere diminishes at the rate of about 6.0° C. per 1,000 meters up to an elevation of 12,000 meters in Europe, or 15,000 meters in the tropics, or even to 20,000 meters over the equator, while above these elevations the temperature is nearly constant or increases a little to the highest levels explored. There have been many conjectures as to the cause of the permanence of the heat of this isothermal region, as overflow of the tropic heat to mid-latitudes, conductional transportation of heat from the lower to the higher levels, production of ozone by the incoming solar radiation in the upper atmosphere and absorption of the short waves of the solar radiation in the same region. There are objections to each of these hypotheses so obvious that we proceed at once to examine the thermodynamic data for at least a statement of the case, if not a complete explanation of the facts.

The computations were executed for the following balloon ascensions, as reported in the volumes:

Europe		*Atlantic Tropics*
Lindenburg,	April 27, 1909 (52°).	Sept. 25, 1907 (35°).
"	May 5, 1909 "	Sept. 9, 1907 (25°).
"	May 6, 1909 "	Aug. 29, 1907 (13°).
"	July 27, 1908 "	July 29, 1907 (13°).
"	Sept. 2, 1909 "	June 19, 1906 (−2°).
Mailand,	Sept. 7, 1906 (45°).	Victoria Nyanza, 1908 (0°).

The mean values are compiled in Table 23, and illustrated in Figs. 5 and 6. Since the data in the isothermal region are not so complete as below it, these results are to be considered as instructive rather than definitive. It will require the work of many years to accumulate and compute the data necessary for normal conditions. The temperatures show that there are as wide local fluctuations in the isothermal region as below it. Furthermore, the temperatures are lowest over the equator, 200°, and gradually increase to 210° in the tropics, or 215° in

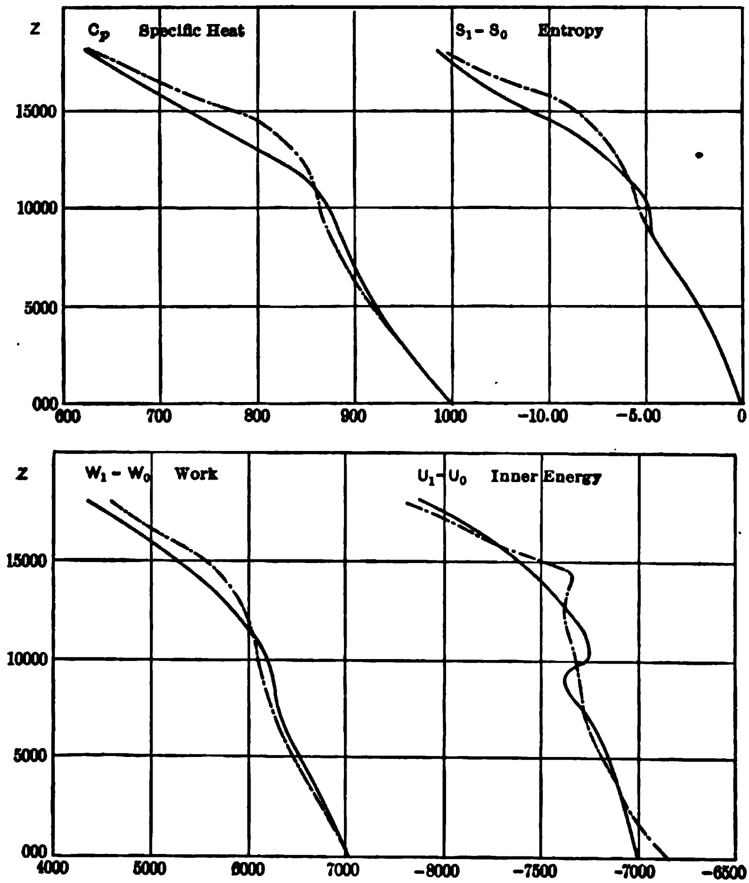
Europe. From these minimum values the temperatures in the isothermal region increase about 10 degrees. These facts appear clearly in the Table 23, and the diagrams, where



Full Line for Europe
Dotted Line for Tropics
FIG. 5

the other data are presented. The data on Fig. 5 (P, $-K$, ρ, $Q_1 - Q_0$) may be considered primary, and those on Fig. 6 (Cp, $S_1 - S_0$, $W_1 - W_0$, $U_1 - U_0$) secondary, as being the machinery of the thermal engine. It is noted that (P, $-K$, ρ)

have one configuration, and (Q, Cp, S, W, U) another configuration. The former is more immediately under the control of gravitation acting downward, and the latter is the result of



Full Line for Europe
Dotted Line for Tropics
FIG. 6

radiation. The circulation q is the balancing governor to the engine which keeps the other two parts in equilibrium.

$(Q_1 - Q_0)$. There is a persistent supply of heat from four conditions: (1) That from the heated earth as the source; (2) that in the lower strata, due to convection within 2,000

TABLE 23

MEAN THERMODYNAMIC VALUES FOR EUROPE

z	T	P	ρ	C_p	q	Q_1-Q_0	S_1-S_0	W_1-W_0	U_1-U	K_{10}
20000	.....	......	......	......	....	......	....	....	......	.....
19000	.....	......	......	......	....	......	....	....	......	.....
18000	.....	......	......	......	....	......	....	....	......	.....
17000	219.5	9141	0.2220	657.15	0.8	−3231	−14.747	4658	−7887	16704
16000	223.4	10983	.2476	689.35	2.5	−2791	−12.620	4962	−7753	18497
15000	221.2	12819	.2764	731.44	3.9	−2392	−11.158	5209	−7602	20083
14000	220.3	14972	.3086	769.99	8.6	−1955	− 8.641	5540	−7494	21850
13000	215.7	17017	.3422	799.60	14.3	−1702	− 7.487	5730	−7431	23883
12000	215.8	19942	.3830	836.65	16.4	−1381	− 6.326	5948	−7329	26548
11000	218.7	23341	.4283	863.20	17.8	−1190	− 5.345	6096	−7285	29953
10000	226.0	27198	.4780	873.16	19.3	−1089	− 4.722	6164	−7253	34502
9000	232.8	31606	.5315	884.82	20.7	−1131	− 4.768	6248	−7379	40630
8000	241.1	36509	.5890	890.49	18.4	−1034	− 4.229	6302	−7336	45275
7000	249.0	41972	.6504	897.75	16.3	− 920	− 3.647	6329	−7249	51345
6000	256.1	48053	.7162	907.35	12.6	− 793	− 3.083	6405	−7198	57729
5000	262.6	54819	.7865	919.62	14.1	− 650	− 2.578	6511	−7160	64352
4000	268.3	62353	.8620	934.08	13.0	− 530	− 1.957	6601	−7131	71739
3000	274.3	70722	.9429	947.41	12.6	− 380	− 1.375	6717	−7097	77430
2000	279.1	80030	1.0299	964.88	11.8	− 241	− 0.857	6825	−7066	86011
1000	284.8	90345	1.1224	979.50	9.2	− 76	− 0.241	6969	−7045	94582
100	289.4	100419	1.2101	993.58	6.1	......	....	....	......	.....

meters of the ground, and involving a supply of latent heat by the condensation of aqueous vapor into water in cloud formation by (295); (3) that in the cirrus cloud region, 9,000 to 15,000 meters elevation, according to the latitude, due to ice formation from frozen water or vapor by (296). In this cirrus region there are other sources of heat supply, such as an accumulation of heat from absorption of radiation producing the new rate of loss of free heat per 1,000 meters. Take the differences in the $(Q_1 - Q_0)$ columns, and the mean values fall into two groups, omitting those in the cirrus layers.

Europe,	12000 to 17000	$\Delta (Q_1 - Q_0)$	$= - 370$	per	1000 meters.
Tropics,	14000 to 17000	"	$= - 476$	"	"
Europe,	1000 to 11000	"	$= - 133$	"	"
Tropics,	1000 to 13000	"	$= - 140$	"	"

TABLE 23

MEAN THERMODYNAMIC VALUES FOR THE ATLANTIC TROPICS

z	T	P	ρ	C_p	q	Q_1-Q_0	S_1-S_0	W_1-W_0	U_1-U_0	K_{10}
20000	.....	......	......	......	...	......	....	....	......	.....
19000	.....	......	......	......	...	......	....	....	......	.....
18000	.....	......	......	......	...	......	....	....	......	.....
17000	227.0	9894	0.2256	668.93	8.0	−3012	−13.359	4911	−7922	17488
16000	217.3	11213	.2513	709.00	7.4	−2570	−10.292	5203	−7774	18647
15000	209.9	13331	.2788	788.87	8.8	−1898	− 8.567	5609	−7506	19328
14000	212.8	15665	.3128	814.91	8.5	−1584	− 7.361	5750	−7334	21544
13000	215.9	18398	.3502	835.84	14.3	−1470	− 6.697	5911	−7381	24731
12000	221.6	21442	.3911	855.58	13.8	−1345	− 5.949	6024	−7369	28286
11000	230.3	24942	.4355	861.08	13.3	−1302	− 5.545	6060	−7362	32588
10000	238.9	28853	.4830	865.62	12.4	−1250	− 5.135	6086	−7336	37288
9000	247.7	33204	.5338	869.26	12.0	−1182	− 4.699	6127	−7309	42938
8000	254.9	38038	.5879	878.51	11.8	−1100	− 4.252	6199	−7298	47822
7000	260.1	43413	.6459	887.63	10.7	−1000	− 3.769	6271	−7271	53935
6000	268.3	49379	.7079	899.98	9.9	− 881	− 3.247	6368	−7248	59582
5000	274.1	56008	.7742	913.55	6.6	− 700	− 2.534	6473	−7173	65980
4000	278.5	63380	.8453	931.88	6.5	− 532	− 1.907	6591	−7123	72304
3000	283.9	71569	.9217	946.73	6.5	− 397	− 1.388	6694	−7091	78606
2000	288.8	80625	1.0031	963.44	5.8	− 187	− 0.641	6846	−7033	86055
1000	292.4	90699	1.0908	984.35	4.4	− 24	− 0.081	6911	−6935	93656
000	299.5	101753	1.1837	993.58	5.8	......	....	....	......	.....

In the Europe group 9,000 to 10,000 is omitted, and in the tropic group 10,000 to 12,000 is omitted, as being regions of special local supply. It appears that heat is lost at about three times greater rate in the isothermal region than in the lower levels. This occurs at the same time the temperature is rising in the isothermal region but falling in the lower levels. The Victoria Nyanza ascensions give $\Delta\,(Q_1 - Q_0) = -144$ throughout the region 3,000 to 18,000, but in this case no inversion of T was found. (4) The principal fact to be explained is the slow rate of loss of heat in the convectional region, 140 per 1,000 meters, as compared with that in the isothermal region, about 400. This is easily accounted for by the following facts: The incoming solar radiation is of short wave lengths, and penetrates to the earth's surface, having only a small amount of selective absorption of radiant energy. This is transformed at the earth into long waves, in changing the temperature energy from 7,000° to 300°, and this heat escapes to space partly by radiation and partly by vertical convection, the latter extending to the isothermal layer, whose height varies with the latitude, and the heat contents of the air at the surface. In the general vertical convection of the atmosphere, as the temperature of a unit mass changes from T_0 to T_1, in the vertical distance $z_1 - z_0$, there is an evolution of heat $Q_1' - Q_0' = Cp_{10} (T_1 - T_0)$, which is added to the atmosphere throughout the convectional region. If $(Q_1 - Q_0)$ is the natural loss of heat by radiation without convection, and $(Q_1' - Q_0')$ the amount evolved by convective cooling of the temperature of the rising mass, then we have, $400 = (Q_1 - Q_0)$ the heat loss in the isothermal region, and $140 = (Q_1 - Q_0) - (Q_1' - Q_0')$ that in the convection region. Hence $260 = (Q_1' - Q_0')$ the heat evolved by vertical motion. This subject will require a fuller development than is at present available, and it is complicated by the fact that the vertical distance through which the mass moves is not well known. The air that has risen by convection with cooling and evolution of heat in one place falls again in other places with heating and absorption of heat. Such places of descending air are, during the night, in the permanent high-pressure belts and in the wandering anti-cyclones. This subject will be sufficiently illustrated in the following sections.

$P, - K$ and ρ.

Pressure, Radiation-Potential, and Density.

The pressure is found to change continuously in a smooth curve from the surface upward, that in the tropics being somewhat higher in value than in Europe. An entirely similar curve is developed by the potential radiation $- K$, the values being always higher than P. The density is also given in a smooth curve, the tropics and Europe crossing at about 7,000 meters. It would seem, then, that the ultimate purpose of T. Q and the other dependent terms is *to so regulate the pressure* *and radiation that they shall change steadily from level to level, under the attraction of the earth's gravitation*, and that all the other thermodynamic values mutually adjust themselves to produce this simple result. Hence, the problem depends upon the rate of loss of $\Delta (Q_1 - Q_0)$ in the lower and the isothermal regions, which is distinctly a physical question. A reason has been already indicated why the rate of transfer of heat should be greater in one region than in the other, and why the isothermal separates from the convectional region.

By (339) we have the following equation:

$$(339)\quad K_{10} = \frac{U_1 - U_0}{v_1 - v_0} = \frac{Q_1 - Q_0}{v_1 - v_0} - P_{10} = \frac{T_{10}(S_1 - S_0)}{v_1 - v_0} \cdot - P_{10}$$

This equation is immediately derived from (259), where $C_T = \left(\frac{\partial Q}{\partial v}\right)_T$ is the latent heat while the temperature is constant; it also comes from (309), where the latent heat r_2 of vaporization is derived by primary analysis; or it may be taken from (338), (339); finally it is found by computation that the last form through the gas coefficients R_a, R_{10} is satisfied, as can easily be verified by the data of Tables 14 and 19. The small discrepancies are due entirely to the velocity term which was eliminated by means of Table 17. $-K_{10}$ is negative in sign because $(U_1 - U_0)$ is negative while $(v_1 - v_0)$ is positive. It seems, then, that the latent heat,

$$(351)\quad C_T = \frac{Q_1 - Q_0}{v_1 - v_0} = K_{10} + P_{10} = P_{10}\left(1 - \frac{R_a}{R_{10}}\right) = P_{10}\left(\frac{R_{10} - R_a}{R_{10}}\right).$$

Hence, the escape of heat or radiation in the earth's atmosphere depends entirely upon the divergence of the gas coefficient R_{10} from the adiabatic value R_a, as was stated. The divergence of the lines P_{10} and $-K_{10}$ in Fig. 5 measures this term. If the velocity is also considered it will be a line near the $-K_{10}$ line, slightly adjusting it to make the pressure transitions gradual. This is the function of the horizontal cloud motions of flowing strata so generally seen in the atmosphere. This confirms the principle of equations (36) to (38), which indicate the relations of pressure, circulation, and radiation to gravitation.

It is easy to see that such data are capable of making all the general thermodynamic formulas (205) to (328), and many others, applicable to the earth's non-adiabatic atmosphere. It should be carefully noted that the density ρ, and the gas coefficient R, must be computed by (176), (177), and not by (175) for R_a constant; that the effective specific heat Cp is variable, and that radiation depends upon this fact. The principal quantities to obtain by observation are the temperature T, and the velocity of circulation q at the height z, and hence the observations for temperature alone, omitting q, are not capable of giving correct radiation data. Finally, the variation of pressure $-dP$ is not proportional to the mass $g\rho dz = g\,dm$, but by (201) the terms $\rho q\,dq + \rho\,dQ$ must be added for circulation and radiation, or else $P = -K$, which is to exclude them from the problem, and reduce it to the unusual adiabatic case. One can now perceive that there is no possibility of solving the general equations of motion in cyclones and anti-cyclones, and in all the other types of circulation, without first eliminating the heat term dQ. Nearly all attempts of meteorologists to solve the circulation problems have been futile chiefly on this account, because of the assumed necessity of ascribing to friction, and to the deflecting force of the earth's rotation on a moving mass, values which they do not actually possess. We shall be able to explain this more fully in the chapter on Dynamic Meteorology, but now proceed to illustrate more at length the thermodynamic terms in other typical conditions of the atmosphere.

The Diurnal Convection and the Semi-diurnal Waves in the Lower Strata

There is a series of problems relating to the semi-diurnal waves observed at the surface, which have been much discussed without satisfactory results, as the semi-diurnal barometric waves and the several electrical and magnetic waves which are associated with the diurnal convection. At the surface the temperature has only a single diurnal wave, and it has not been possible to match these two series of data in a definite relation of cause and effect. The difficulty in studying the general problem has been the lack of observations in the free air above the surface, especially during the night. The only exception to this defect is the series of kite ascensions at the Blue Hill Observatory, 1897–1902, which were discussed in my papers, *Monthly Weather Review*, February to August, 1905. In these it was shown that the single diurnal temperature wave changes into a semi-diurnal wave at about 400 meters above the surface, and that the semi-diurnal waves die away within two or three thousand meters of the ground. It was also indicated that the other data, namely, vapor pressure, atmospheric electric potential, ionization, and magnetic fields, have diurnal variations closely matching the diurnal circulation. This section gives the result of another discussion of the subject, using the data of the Cordoba and Pilar stations, Argentina.

There are two hypotheses regarding the origin of the semi-diurnal barometric waves: (1) The forced oscillation of the entire atmosphere, as proposed by Lord Kelvin, and developed by Margules, Hann, Jaerisch, Gold, and others; (2) the effect of the diurnal convection, proposed in general terms by Espy, and studied by Ferrel, Köppen, Sprung, Bigelow, and others. The important objections to the former hypothesis are, that these waves do not embrace the entire atmosphere and are usually confined within 2,000 meters of the surface; that the analytical equations and harmonic analyses merely represent, in other forms, the data assumed for their coefficients, and do not reach the origin of the physical causes; and that these equations have not the radiation and heat terms, which are more important than the friction and the deflecting force of rotation.

The method of treating the Cordoba-Pilar data was to assume a uniform temperature T, and pressure P, on the 3,500-meter level, and then, by studying the observed temperatures and wind velocities on several levels, 000, 200, 400 . . . 3,500 meters, at

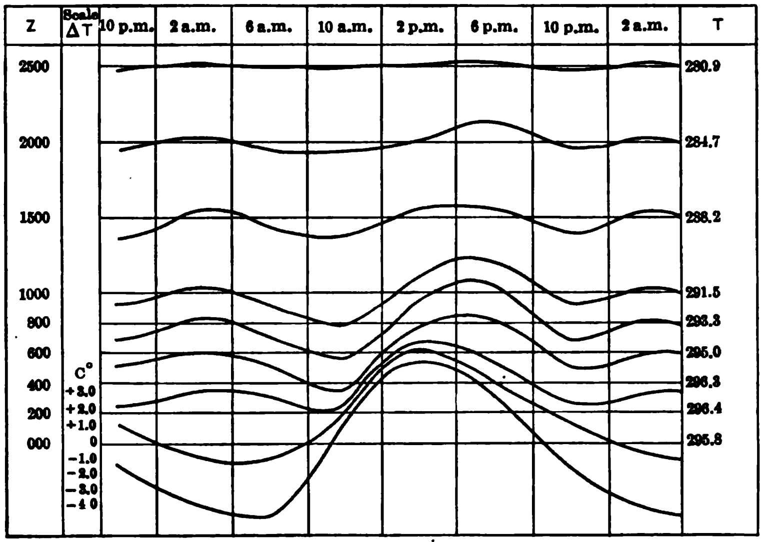


FIG. 7. Δ T—Temperature of the semi-diurnal waves above 400 meters and the diurnal wave near the surface
T is the mean temperature on the given level.

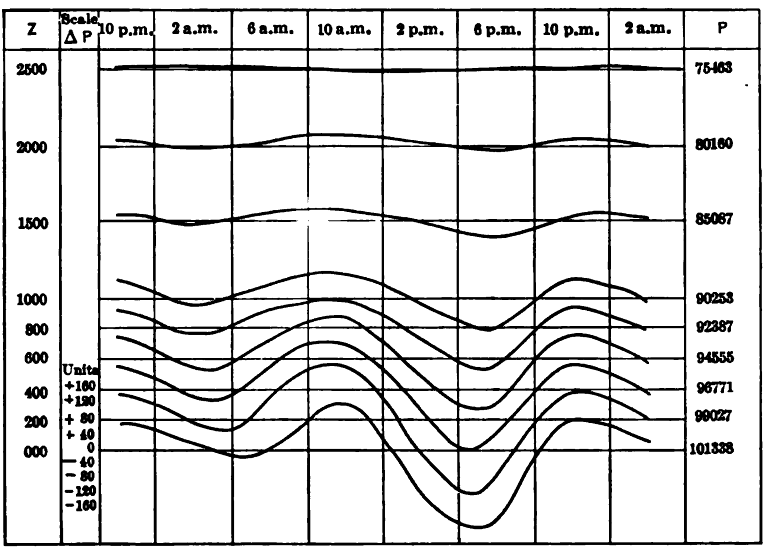


FIG. 8. Δ P—Pressure of the semi-diurnal waves in all these strata vanishing at about the level 2500 meters

the hours (2, 6, 10) A.M., (2, 6, 10) P.M., proceed by computations entirely similar to those of Tables 14, 17, 19, to derive a pressure P_0 at the surface which would exactly match the mean

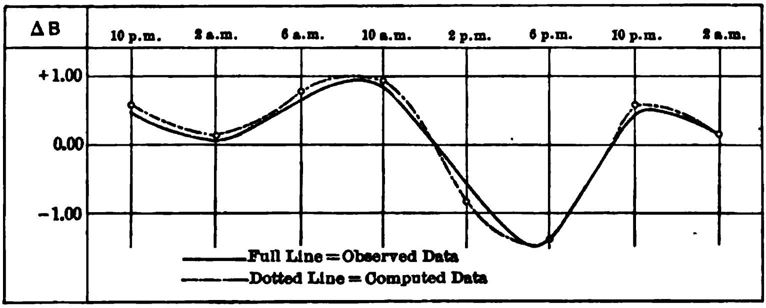


FIG. 9. Computed and observed pressure waves at the surface

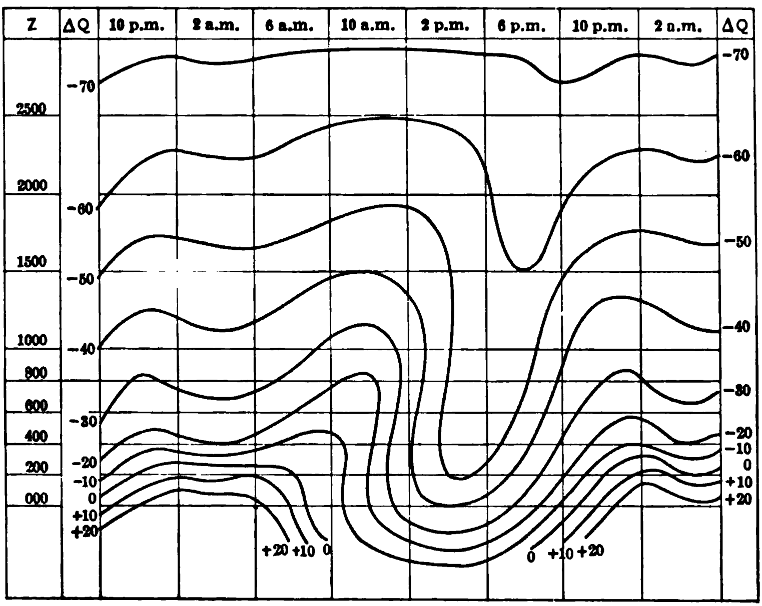


FIG. 10. The loss of heat for every 200 meters $\Delta Q = (Q_1 - Q_0)$

pressure as observed at Cordoba. It required several trial computations to accomplish this result, especially during the night hours where direct observations were lacking above the surface, and the outcome is indicated in Tables 24 and 25, and in Figs. 7, 8, 9, 10. All the other data, as computed, will be found in Bulletin No. 3, O. M. A. Transferring the data of Table 24 to the diagrams, it is seen that the single diurnal temperature wave at the surface transforms into semi-diurnal waves at 400 meters, and that these die away at 2,500 meters. This result confirms the discussion of the Blue Hill data in all respects. The corresponding pressure waves are semi-diurnal throughout these strata from the surface to 2,500 meters, and there they vanish. The night wave is weaker than the day wave in consequence of the temperature inversion near the surface, both waves being nearly equal at 1,000 meters.

Since the barometric variations are alone of interest, the base line has been taken as that near sea level. The discrepancy at 2 P.M. is due to an imperfect temperature distribution at that hour, which should be made cooler by a few tenths of a degree.

By adjusting the vertical temperatures a little it would be possible to reproduce the observed curve with precision through the non-adiabatic computations. There are no other temperatures which could reproduce this pressure wave at the surface, and this fact is proof of the cause of the observed pressure system. In order to understand more fully the origin of the temperature system, the values of $\Delta\ (Q_1 - Q_0)$, the variations on the daily mean are plotted on Fig. 10 and the curves of equal heat losses are drawn. The data are somewhat imperfect, but the general result is not doubtful. It shows that there are two principal axes of heat exchange, that in the afternoon, 2 P.M., at the surface, to 8 P.M., at 2,500 meters, and that in the night from 1 A.M at 2,500, to 4 A.M., at the surface. That is to say, the air rises obliquely in the afternoon to the right and falls in the night, also to the right. The air rises and falls in such a zigzag path as gives a turning-point at about 10 P.M. above and 10 A.M. below, judging by the crests. The rising air cools by expansion, and the falling air heats by compression, the former producing the afternoon wave and the latter the night wave to within 400 meters of the surface on Fig. 7. At this level the more rapid cooling of the ground during the night makes itself felt, and there is radiation

TABLE 24

THE SEMI-DIURNAL TEMPERATURE WAVES IN THE STRATA 400 TO 2,500 METERS, TOGETHER WITH THE SINGLE DIURNAL TEMPERATURE WAVE AT THE SURFACE

z	T	P	q	$-(Q_1-Q_0)$	T	P	q	$-(Q_1-Q_0)$
	2 A.M.				6 A.M.			
2500	281°.0	75464	3.0	+151.8	280°.8	75466	6.0	+140.2
2000	285. 0	80153	4.0	+131.9	284 .2	80162	7.0	+120.1
1500	288 .6	85080	5.0	+111.9	287 .5	85104	7.0	+ 97.6
1000	291 .9	90230	5.0	+ 38.5	290 .6	90288	6.0	+ 34.9
800	293 .6	92368	5.0	+ 40.2	292 .3	92428	5.0	+ 33.1
600	295 .0	94530	4.0	+ 30.0	294 .3	94604	4.0	+ 25.8
400	295 .8	96743	4.0	+ 16.1	295 .4	96820	4.0	+ 13.2
200	293 .7	99018	3.0	− 11.7	293 .4	99095	3.0	− 15.8
000	231 .7	101349	2.0	....	291 .0	101440	2.0	....
	10 A.M.				2 P.M.			
2500	280°.7	75462	9.4	+144.2	280°.9	75462	9.7	+146.7
2000	284 .0	80173	9.4	+114.2	284 .8	80162	10.3	+133.0
1500	287 .0	85112	9.4	+ 87.4	288 .8	85084	10.6	+133.0
1000	289 .5	90313	8.3	+ 23.0	292 .6	90228	9.2	+ 49.2
800	291 .0	92460	8.1	+ 19.7	294 .8	92355	8.7	+ 49.2
600	292 .5	94656	7.9	+ 18.9	296 .9	94510	8.3	+ 49.9
400	294 .5	96886	7.6	+ 16.2	299 .0	96703	7.9	+ 52.2
200	296 .2	99154	6.0	+ 23.4	300 .5	98930	7.0	+ 52.2
000	297 .7	101459	4.0	....	301 .3	101214	5.0	....
	6 P.M.				10 P.M.			
2500	281°.5	75458	8.0	+159.3	280°.8	75466	5.0	+142.0
2000	286 .0	80142	9.0	+156.5	284 .3	80170	6.0	+117.6
1500	290 .0	85038	9.0	+145.6	287 .1	85102	6.0	+ 88.5
1000	293 .5	90166	8.0	+ 51.9	290 .7	90294	6.0	+ 34.9
800	296 .0	92278	8.0	+ 58.9	292 .2	92432	5.0	+ 28.0
600	297 .3	94426	7.0	+ 51.6	294 .2	94606	5.0	+ 29.0
400	298 .0	96636	6.0	+ 42.1	295 .1	96835	4.0	+ 19.9
200	298 .6	98867	5.0	+ 31.6	295 .7	99095	3.0	+ 3.2
000	299 .0	101149	4.0	....	294 .3	101414	2.0	....

from the descending air to the ground, and an inversion of temperature under ordinary circumstances. The friction and the earth's deflection have very little influence on the temperature and pressure conditions, and the circulation cannot be studied by itself until the radiation or heating terms have been eliminated.

TABLE 25

THE CORRESPONDING VALUES OF $B_c = P/g_0 \rho_m$ ON THE LEVEL $z = 000$ METERS ARE NOW GIVEN, AND COMPARED WITH THE OBSERVED VALUES OF B_0 BY MEANS OF THE DIFFERENCES

Surface	2 A.M.	6 A.M.	10 A.M.	2 P.M.	6 P.M.	10 P.M.	Mean
B_c	760.20	760.88	761.03	759.18	758.70	760.70	760.12
B_0	760.16	760.75	760.98	759.52	758.71	760.59	760.12
ΔB_c	+0.08	+0.76	+0.91	−0.94	−1.42	+0.58	......
ΔB_0	+0.04	+0.63	+0.86	−0.60	−1.41	+0.47	......

These curves can be reduced to the harmonics if desired. The value of the radiation exponent is $a = 3.82$ throughout the twenty-four hours.

The Thermodynamic Structure of Cyclones and Anticyclones

There has been much speculation regarding the forces that generate the powerful circulations in storms known by the name of cyclones and anticyclones, or low-pressure and high-pressure areas respectively. These will be more fully mentioned in the chapter on Dynamic Meteorology, but here we proceed to apply the principles just illustrated in the diurnal convection. From numerous kite and balloon ascensions in all parts of these local circulations, it has been learned what is the usual distribution of the temperature, and from the cloud observations what is the direction and velocity of the wind motion or "vector" in all areas, and all altitudes up to at least 10,000 meters. Compare the International Cloud Report, 1898, the *Monthly Weather Review*, January to July, 1902, April to June, 1904, January to August, 1906, October, 1907, to February, 1909, also the daily Weather Synoptic Charts, for numerous studies and details. From these data we have selected the temperatures T, and velocities q, given in Tables 26, 25, and Figs. 11, 12. The computed values of the pressure P, and the free heat $(Q_1 - Q_0)$ are given in the same tables and figures, while the other thermodynamic data are summarized in Bulletin No. 3, O. M. A. Only three diagrams are extracted from the tables, but these are enough to

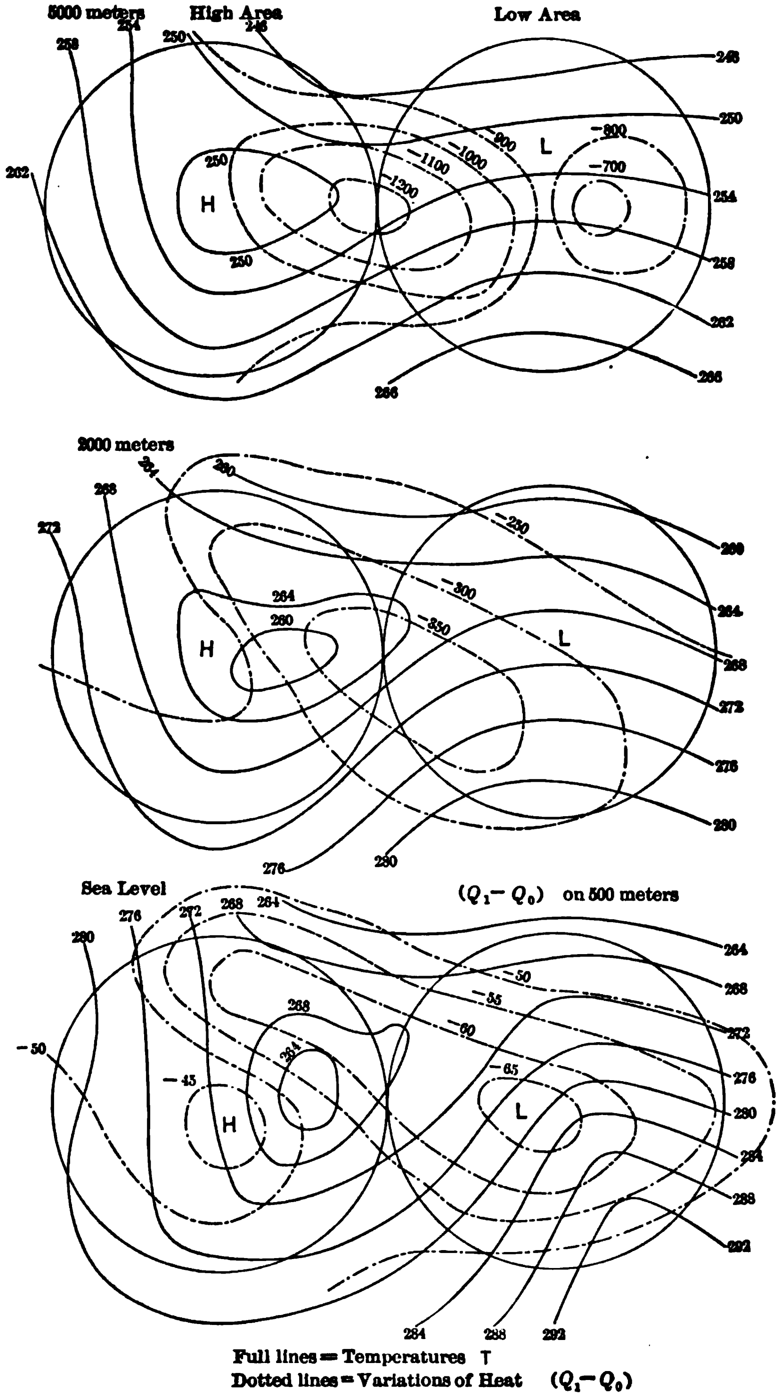


FIG. 11. The temperatures and heat variations in high and low areas

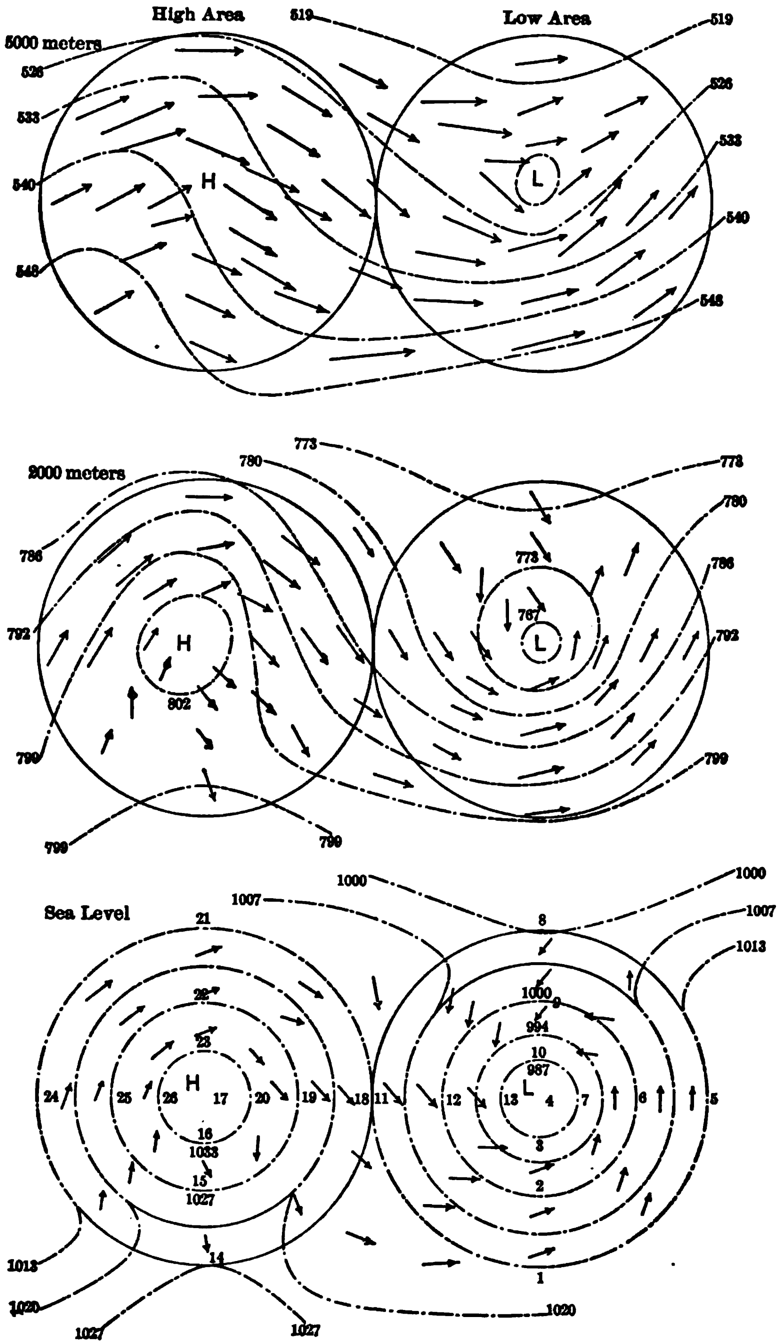


FIG. 12. Pressures and wind velocities in high and low areas

TABLE 26

SOME VALUES OF T. P. q $(Q_1 - Q_0)$ IN CYCLONES

TEMPERATURE T

A	C	B	000	500	1000	1500	2000	2500	3000	4000	5000
1	*S.*	760	290.0	289.0	288.0	286.3	284.3	282.1	279.8	274.5	268.0
2	*S.*	750	287.5	286.4	284.5	282.7	280.4	278.0	276.0	270.5	264.0
3	*S.*	740	284.0	282.6	280.8	279.0	276.8	274.4	271.6	266.0	260.0
4	*C.*	735	280.0	279.0	277.5	275.5	273.2	271.0	269.0	263.0	257.0
5	*E.*	760	278.0	276.6	274.8	272.4	269.7	267.0	264.5	259.5	255.0
6	*E.*	750	281.0	279.3	277.0	274.6	272.0	269.5	267.0	261.7	256.0
7	*E.*	740	283.5	281.4	279.1	276.6	274.0	271.0	268.0	263.0	257.0
8	*N.*	750 I	266.0	265.8	264.5	263.0	261.5	259.5	257.0	250.0	243.0
9	*N.*	750 II	272.5	271.2	269.7	268.0	266.0	263.9	261.7	256.0	249.0
10	*N.*	740	277.0	276.0	274.8	273.0	270.8	268.5	266.0	260.5	254.0
11	*W.*	760	268.0	267.5	266.8	266.0	265.0	264.0	262.0	258.0	253.0
12	*W.*	750	273.0	272.6	272.0	271.0	269.8	268.0	266.0	261.0	255.0
13	*W.*	740	277.0	276.0	275.0	273.7	272.0	270.0	268.0	263.0	256.5

PRESSURE P

A	C	B	000	500	1000	1500	2000	2500	3000	4000	5000
1	*S.*	760	101322	95518	90010	84814	79893	75208	70778	62566	55161
2	*S.*	750	99988	94194	88724	83548	78628	73958	69540	61367	54000
3	*S.*	740	98655	92873	87418	82230	77340	72683	68278	60127	52801
4	*C.*	735	97990	92158	86672	81482	76565	71908	67498	59366	52051
5	*E.*	760	101322	95275	89562	84140	78997	74124	69515	61016	53424
6	*E.*	750	99988	94064	88462	83150	78114	73338	68812	60467	52993
7	*E.*	740	98655	92864	87380	82166	77223	72532	68072	59861	52486
8	*N.*	750 I	99988	93706	87884	82372	77165	72278	67641	59114	51465
9	*N.*	750 II	99988	93890	88142	82708	77587	72747	68163	59787	52179
10	*N.*	740	98655	92747	87176	81896	76915	72187	67720	59481	52080
11	*W.*	760	101322	95067	89220	83654	78460	73582	68911	60404	52860
12	*W.*	750	99988	93983	88252	82868	77790	72997	68477	60150	52690
13	*W.*	740	98655	92745	87156	81890	76930	72223	67779	59608	52256

VELOCITY q

A	C	B	000	500	1000	1500	2000	2500	3000	4000	5000
1	*S.*	760	4.1	6.5	9.0	13.9	21.0	26.5	27.0	24.0	23.0
2	*S.*	750	4.9	9.0	12.0	15.4	21.2	27.0	26.0	24.0	26.0
3	*S.*	740	6.5	11.0	14.0	16.9	22.8	30.0	34.0	34.0	33.0
4	*C.*	735	7.0	11.0	15.0	19.0	30.0	40.0	42.0	40.0	39.0
5	*E.*	760	5.2	8.6	12.4	16.0	19.8	23.0	24.0	25.0	26.0
6	*E.*	750	5.0	9.0	13.0	17.0	22.0	26.0	27.0	27.0	27.0
7	*E.*	740	5.0	7.5	11.0	15.0	22.0	29.0	29.0	29.0	28.0
8	*N.*	750 I	3.5	5.0	5.8	6.0	7.0	7.5	8.0	8.0	7.5
9	*N.*	750 II	4.5	6.0	8.0	11.0	14.5	18.0	20.5	24.5	25.0
10	*N.*	740	6.0	8.0	11.0	15.0	18.0	22.0	26.0	31.0	33.0
11	*W.*	760	5.0	8.0	12.0	17.0	22.0	26.0	27.0	28.0	31.0
12	*W.*	750	6.0	9.0	13.0	19.0	24.0	27.0	28.0	30.0	33.0
13	*W.*	740	6.0	10.0	14.0	20.0	25.0	29.0	32.0	34.0	37.0

FREE HEAT $(Q_1 - Q_0)$

1	*S.*	760	...	−45.9	−118.4	−209.7	−333.0	−386.3	−313.4	−789.6	−872.3
2	*S.*	750	...	−61.2	−124.5	−189.3	−297.2	−374.2	−253.8	−631.9	−861.6
3	*S.*	740	...	−69.2	−125.2	−186.0	−316.2	−425.8	−408.3	−659.0	−764.7
4	*C.*	735	...	−70.5	−150.6	−221.4	−465.6	−597.1	−377.6	−625.6	−810.9
5	*E.*	760	...	−54.5	−129.3	−189.8	−248.3	−282.7	−283.5	−671.9	−851.9
6	*E.*	750	...	−55.7	−122.8	−184.1	−263.8	−303.7	−276.4	−621.4	−773.4
7	*E.*	740	...	−55.0	−104.1	−167.4	−286.0	−377.2	−232.2	−571.8	−695.6
8	*N.*	750 I	...	−50.1	−124.3	−184.7	−251.3	−304.7	−352.3	−791.3	−890.2
9	*N.*	750 II	...	−40.6	−109.8	−183.5	−254.0	−316.8	−356.8	−828.4	−874.3
10	*N.*	740	...	−48.6	−130.3	−213.5	−261.7	−338.5	−398.1	−864.0	−919.6
11	*W.*	760	...	−59.7	−158.1	−260.1	−360.9	−427.8	−417.6	−960.4	−1204.0
12	*W.*	750	...	−62.5	−167.1	−286.6	−364.4	−392.4	−394.8	−925.4	−1110.3
13	*W.*	740	...	−66.7	−151.7	−271.5	−341.0	−389.2	−417.7	−860.9	−1041.4

TABLE 27

SOME VALUES OF $T.\ P.\ q\ (Q_1 - Q_0)$ IN ANTICYCLONES

TEMPERATURE T

A	*C*	*B*	000	500	1000	1500	2000	2500	3000	4000	5000
14	*S.*	770 I	278.0	277.4	276.0	274.6	273.0	271.0	268.6	264.0	259.0
15	*S.*	770 II	273.5	273.0	272.0	270.4	268.7	267.0	265.0	260.0	254.0
16	*S.*	775	272.0	271.0	269.5	267.8	266.0	264.0	262.0	256.5	250.0
17	*C.*	777	272.0	271.0	269.2	267.0	264.5	262.0	259.5	254.5	248.0
11 = 18	*E.*	760	268.0	267.5	266.8	266.0	265.0	264.0	262.0	258.0	253.0
19	*E.*	770	264.0	263.8	262.7	262.0	261.2	259.7	258.0	254.0	249.5
20	*E.*	775	266.0	265.0	264.0	262.9	261.5	259.8	258.0	254.0	249.0
21	*N.*	760	271.0	269.8	268.2	266.3	264.2	262.0	259.4	254.0	248.0
22	*N.*	770	272.0	271.4	270.3	269.1	267.6	265.4	262.6	257.0	251.0
23	*N.*	775	272.0	271.0	269.5	267.6	265.5	263.1	260.4	255.0	249.5
24	*W.*	760	282.0	281.2	280.3	279.1	277.7	276.0	273.8	268.0	262.0
25	*W.*	770	278.0	277.1	276.0	274.7	273.0	271.0	268.5	263.0	257.0
26	*W.*	775	274.0	273.3	271.8	270.0	268.0	265.9	263.7	258.0	252.0

PRESSURE P

14	*S.*	770 I	102655	96583	90794	85330	80146	75292	70677	62160	54551
15	*S.*	770 II	102655	96594	90550	85028	79805	74865	70220	61644	53970
16	*S.*	775	103322	97020	91080	85466	80164	75157	70435	61741	53949
17	*C.*	777	103588	97270	91316	85670	80343	75289	70520	61740	53888
18	*E.*	760	101322	95067	89220	83654	78460	73532	68911	60404	52860
19	*E.*	770	102655	96232	90184	84492	79144	74122	69392	60711	53016
20	*E.*	775	103322	96906	90818	85126	79753	74698	69928	61187	53419
21	*N.*	760	101322	95110	89266	83736	78512	73577	68913	60321	52646
22	*N.*	770	102655	96394	90676	85088	79857	74893	70197	61549	53803
23	*N.*	775	103322	97020	91080	85472	80153	75142	70395	61656	53845
24	*W.*	760	101322	95376	89716	84410	79394	74638	70136	61824	54349
25	*W.*	770	102655	96563	90772	85312	80146	75278	70648	62111	54476
26	*W.*	775	103322	97073	91176	85596	80327	75357	70647	61974	54203

VELOCITY q

14	*S.*	770 I	5.0	7.0	10.0	12.0	16.0	19.0	20.0	21.0	22.0
15	*S.*	770 II	5.0	7.0	8.0	9.0	12.0	15.0	16.0	17.0	18.0
16	*S.*	775	3.0	5.0	7.0	8.0	9.0	11.0	12.0	14.0	16.0
17	*C.*	777	4.0	5.0	6.0	8.0	9.0	10.0	11.0	12.0	14.0
18	*E.*	760	5.0	8.0	12.0	17.0	22.0	26.0	27.0	28.0	31.0
19	*E.*	770	5.0	7.0	10.0	14.0	19.0	23.0	25.0	26.0	28.0
20	*E.*	775	4.0	6.0	9.0	12.0	16.0	19.0	20.0	22.0	24.0
21	*N.*	760	5.0	8.0	11.0	14.0	18.0	22.0	24.0	26.0	27.0
22	*N.*	770	6.0	9.0	13.0	16.0	20 0	23.0	26.0	28.0	30.0
23	*N.*	775	4.0	7.0	10.0	13.0	17.0	21.0	23.0	24.0	26.0
24	*W.*	760	5.0	8.0	10.0	13.0	14.0	15.0	16.0	17.0	18.0
25	*W.*	770	5.0	7.0	9.0	11.0	12.0	13.0	14.0	15.0	16.0
26	*W.*	775	4.0	6.0	8.0	10.0	12.0	12.5	13.0	14.0	15.0

FREE HEAT $(Q_1 - Q_0)$

14	*S.*	770 I	......	−49.7	−131.4	−189.0	−282.0	−337.1	−350.6	−804.4	−977.1
15	*S.*	770 II	......	−51.8	−121.5	−186.1	−266.1	−331.1	−359.0	−837.1	−988.0
16	*S.*	775	......	−43.3	−113.2	−167.7	−224.8	−289.5	−332.2	−794.6	−934.0
17	*C.*	777	......	−39.8	−104.0	−164.6	−204.8	−248.9	−293.1	−706.1	−868.1
18	*E.*	760	......	−59.7	−158.1	−260.1	−360.9	−427.8	−417.6	−960.4	-1204.5
19	*E.*	770	......	−51.0	−142.8	−242.8	−352.0	−420.2	−442.4	−972.2	-1194.7
20	*E.*	775	......	−45.9	−130.2	−209.3	−299.6	−356.1	−379.1	−918.7	-1109.7
21	*N.*	760	......	−53.2	−125.7	−191.5	−270.1	−335.3	−346.0	−770.4	−895.7
22	*N.*	770	......	−61.3	−154.6	−219.9	−311.0	−356.9	−408.2	−834.8	−980.2
23	*N.*	775	......	−51.8	−126.7	−192.9	−270.2	−333.5	−343.4	−740.4	−923.1
24	*W.*	760	......	−55.2	−123.9	−206.7	−246.8	−304.4	−355.1	−808.8	−942.4
25	*W.*	770	......	−47.2	−119.8	−188.3	−238.6	−292.2	−338.8	−780.6	−921.9
26	*W.*	775	......	−47.7	−119.5	−181.1	−238.1	−272.5	−321.0	−762.8	−903.0

indicate clearly the principles that are involved in their structure. Continuing the computations to higher levels, it is found that the temperature lines or isotherms, the pressure lines or isobars, and the velocity lines or vectors, coincide at every point in direction. In the lower levels these lines cross each other at various angles in the areas marked 1 to 26 on the sea level of Fig. 12, which shows the order of the computations: (1) The upper undisturbed circulation, where *T. P. q* coincide, belongs to the general circulation, and (2) the lower disturbed circulation, where *T. P. q* do not coincide, to the combined general and local circulation; (3) the purely local circulation, the cyclone and anticyclone proper, can be separated from the second by vector composition, since (2) is the resultant of (1) and (3). It is found that the disturbing circulation (3) is similar in configuration to that at sea level up to 10,000 meters, if it penetrates the general eastward drift so high, that it usually increases in intensity to the 3,000-meter level, and then gradually dies out as the head is stripped away in the rapidly flowing upper currents. There is no evidence of a change in the type of the circulation, and therefore the physical origin of the structure of a cyclone is the same throughout. In the International Cloud Report it was shown that a series of warm currents from the south interlock alternately with another series of cold currents from the north, in the United States and adjacent regions, and that these local circulations are the mixing regions where the interchange of the temperature goes on toward a thermodynamic equilibrium under the force of gravitation. Fig. 11 shows the temperature distribution T and the free-heat distribution $(Q_1 - Q_0)$. The temperature is deflected to the south on the east side of the high area, and to the north on the east side of the low area. The deflections of the isotherms diminish with the altitude and finally disappear as these disturbing currents diminish in strength. There are no cold-center anticyclones, and no warm-center cyclones, as has been assumed in many theoretical discussions. The distribution of $(Q_1 - Q_0)$ is in elliptical figures whose centers are on the border of the high and low areas, and they show where the exchange of heat is going on most vigorously. The radiation heat increases with the height in consequence of the general radiation of the atmosphere increasing upward. It would be well to separate the purely local $(Q_1 - Q_0)$ from the general as can be done by computation. The pressures of Fig. 12 depend entirely upon the temperature assigned to the several areas, and not upon the circulation, the deflecting force, the centrifugal force, or the friction, or any other minor condition. The air column, though temporary in position at a given instant, presses upon the level of computation, in consequence of the air masses which are determined only by the density, since this in turn is a function of the temperature. The given temperature structure must be continuously renewed by circulation of air from the warm and cold regions, or else the gravitation would soon flatten down all the disturbed temperature and pressure levels. The ultimate source of heat is the sun's insolation, chiefly on the tropics, and radiation in general from the atmosphere. The tropics are the boiler and the polar regions the condenser of the thermal engine, and the cyclones and anticyclones are the working machinery of motion. The general circulation depends upon the heat of the tropics, with westward drift to the south of 33° latitude, and eastward drift in middle latitudes from 33° to 66° latitude, that in the polar zone being irregular. There are, however, centers of general action along the high-pressure belt separating the westward from the eastward drifts, such that there are leakage currents from one of these zones to the other. There are such centers of action over the tropic north and south Atlantic and Pacific Oceans, those in the same hemisphere being broken through by the western and eastern continents. Such leakage currents flow northward from the Gulf of Mexico over the United States, and from the north Pacific upon the northwestern States in a southward direction. The interflowing of these two series of warm and cold currents upon the United States and Canada is the immediate cause of the numerous cyclones and anticyclones that wander eastward over this region. Forecasts are made of the probable detailed action of the weather conditions in all areas, as learned by experience with the types that these local circulations assume. The operation of the several elements, temperature, pressure, wind, and precipitation, is very complex and irregular, so that practical forecasts are difficult and uncertain, except in the cases of vigorously developed storms, which move along paths quite well determined by the pressure and temperature distributions.

It should be noted in Fig. 12 that there is a saddle of higher pressure to the north of the center in low areas, and to the south of high areas there is a saddle of low pressure. These gradually diminish with the height and usually disappear above the 3,000-meter level. It can be seen that the winds on the sea level generally blow out of a high area into a low area, by curves having reversed spiral forms crossing the isobars at angles varying with the place. The winds flow more closely along the isobars at higher levels, as stated, and from 5,000 meters to

10,000 meters it would be safe to draw the isobars and isotherms from the wind directions as observed in the high clouds. It follows that high-level pressure and temperature charts are the true indicators of the general movement of storms across the continent, because they show the direction of the eastward drift when the isobars on the sea level do not clearly indicate it. Such charts were prepared by the author for the sea level, the 3,500-foot plane, and the 10,000-foot plane for the United States, and they proved to be most instructive for the public forecast service.

Attention is called to the fact that the isobars are all marked in the notation of Table 1, units of force $B = P/100$, and the isotherms in absolute temperatures T. Thus, we have for the pressure:

P	B	Mercury mm.
103322	1033.2	775.0
102655	1026.5	770.0
101322	1013.2	760.0
99988	999.9	750.0
98655	986.6	740.0
97990	979.9	735.0

P is the dynamic pressure in the M. K. S. System; B is this pressure divided by 100 for practical use; *mm.* is the millimeters of mercury of a barometer. B and *mm.* are related very closely in the ratio 4 to 3. Hence, by making the scale of a barometer in divisions each three-fourths of a millimeter, it would be only necessary to multiply the reading by 100 to obtain the dynamic pressures useful in all computations on the dynamics and thermodynamics of the atmosphere. This change in units is so simple, and so far-reaching in its beneficial results, that it is strongly recommended to meteorologists. Generally, a full set of Tables should be constructed to supersede the mercurial British and the Metric systems now in use.

Further attention is called to the fact that these computations fully satisfy equation (196) in a non-adiabatic atmosphere, and that, therefore, the author's theory of the non-asymmetric cyclone and anticyclone, due to interflowing currents at different temperatures subject to the attraction of gravitation at every point, is fully verified, since the computed and the observed values are in agreement.

The Planetary Circulation and Radiation. The Observations of Temperature and Velocity

The greatest difficulty in discussing the problems of the planetary circulation and radiation consists in determining the proper temperatures and velocities of the circulation in all latitudes from the equator to the pole, and at all altitudes from the surface up to the practical limit of the balloon ascensions, as 30,000 meters. The number of available observations is very limited throughout the tropics, they are lacking entirely in the arctic zone, and above 14,000 meters in the isothermal region they are insufficient for our purposes. In spite of these difficulties it has been thought proper to execute the extensive computations, for the sake of the general instruction regarding various unsolved problems of meteorology, which depend upon such data. There are several accessible reports and compilations on the results of balloon ascensions, and we utilize them without further references: Rykachef for Russia, Dines for England, Teisserenc de Bort for France, Wegener for Germany, Rotch for St. Louis, Teisserenc de Bort and Rotch for the Atlantic Ocean, Berson for Victoria Nyanza and East Africa. Table 27 contains a summary of the original mean observations arranged according to the latitude, and Table 30 contains the adopted temperature system, which fairly represents this type of distribution. An inspection of these original temperatures presents a great difficulty when they are compared with the wind velocities and directions in the tropics. It is seen that there is a decrease of temperature in the convectional region from the equator to the pole, except in the low levels of the tropics, as indicated in Fig. 13, Case II. When the temperature rises towards the pole there is westward wind, as in the trades of the tropics; when the temperature falls toward the pole there is eastward drift, as in the temperate zones. This was first developed by Bigelow, 1904, and confirmed by De Bort and Rotch in their report, 1909, thus establishing a fundamental property of all atmospheric motions as indicated by Helmholtz. The trades blow steadily westward at a moderate velocity, while the eastward drift reaches a mean velocity of about 35 meters per second

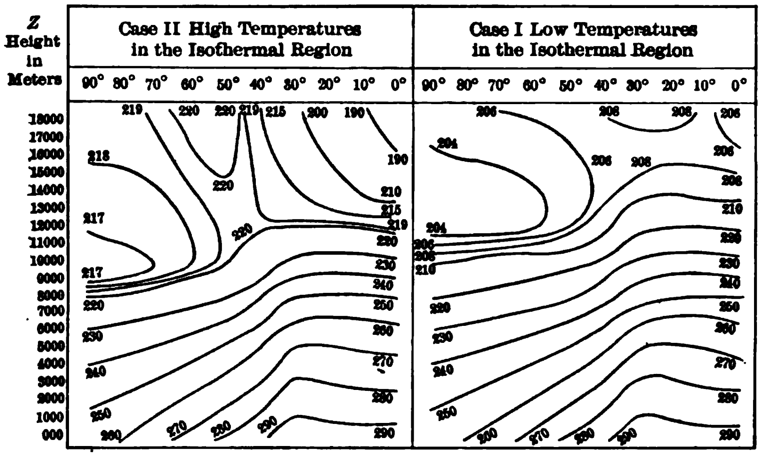


FIG. 13. Two typical cases of the observed temperatures in the earth's atmosphere up to 19,000 meters

in middle latitudes, where the temperature falls rapidly toward the pole at high elevations.

These cases illustrate the movement of a temperature maximum from the tropics into the temperate zones of the isothermal region.

The general questions of temperature are greatly complicated by the necessity of adapting them to the observed velocities, and for those the observations are too limited in number in the isothermal region and in the arctic zone to be decisive. Table 33 contains one system of velocities, which conform to the adopted temperatures of Table 30. The velocities are directed westward in the convection region of the tropics, with maximum on the 4,000-meter level, minimum at the 13,000-meter level, and a region of alternately westward and eastward velocities above that level, except immediately over the equator, where the wind is steadily westward. In the latitudes on the poleward side of the high-pressure belt, which is in latitude 30°,

TABLE 27

THE MEAN OBSERVED TEMPERATURES IN THE ATMOSPHERE FROM BALLOON ASCENSIONS ARRANGED IN THE ORDER OF LATITUDE

	Russia	England	Germany	France	St. Louis	Atlantic Ocean Tropics				Vict. Nyanza
Latitude	56°	53°	51°	49°	39°	35°	25°	15°	5°	0°
Number	143	200	380	581	23	12	6	8	6	12
19000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
18000	.....	.....	.....	.....	.....	.....	.....	.....	.....	190.5
17000	.....	.....	.....	.....	.....	.....	.....	.....	.....	197.1
16000	.....	.....	.....	.....	.....	.....	.....	.....	.....	202.6
15000	.....	.....	.....	.....	218 3	.....	.....	.....	.....	206.8
14000	.....	219.1	218.6	218.9	212.2	211.2	210.7	210.0	214.1	210.8
18000	.....	219.3	218.5	218.6	214.2	214.6	212.6	216.2	218.1	216.0
12000	218.3	219.6	218.8	217.8	216.7	219.4	217.4	223.6	224.7	222.6
11000	217.7	219.4	220.2	219.0	221.0	225.8	224.7	231.8	231.8	231.4
10000	218.9	223.1	223.4	223.7	226.2	233.7	233.2	240.2	239.3	238.9
9000	224.3	228.4	228.6	229.5	232.9	242.0	241.8	248.2	247.2	246.1
8000	231.4	235.2	235.0	236.8	239.8	250.0	249.6	255.5	254.6	250.7
7000	239.1	241.8	242.2	244.0	248.5	257.6	256.6	261.8	261.4	258.0
6000	246.4	248.8	249.3	251.1	256.1	264.4	263.0	267.6	267.3	263.4
5000	253.1	255.4	256.1	257.6	262.7	270.5	268.8	272.8	272.1	269.2
4000	259.3	261.7	262.3	263.6	268.6	275.8	274.6	277.9	277.2	274.7
3000	265.8	268.8	268.0	269.0	273.5	280.9	279.3	282.8	282.0	280.8
2000	269.9	272.5	273.1	273.7	277.8	285.4	283.9	287.5	286.7	288.4
1000	274.3	277.1	277.6	278.3	281.1	290.9	288.7	292.2	292.3	296.2
000	277.1	281.3	282.9	282.5	285.9	298.9	296.2	298.4	300.8	.....

as given in Table 29 for all elevations, the wind is eastward, increasing from the surface to a maximum on the 9,000–10,000-meter level, where it suddenly falls in velocity, and prevails eastward or variable throughout the isothermal region. These velocity conditions conform to Bigelow's observations at Washington, D. C., 1896–97, International Cloud Report, Charts 11, 14, and *Monthly Weather Review*, April, May, June, 1904. The reports of the International Committee show that for the hemisphere at large, these westward and eastward circulations tend to concentrate about "centers of action," wherein the continuity of the high-pressure belt around the globe in longitude is broken up into sections, one over the Atlantic Ocean in each hemisphere, and another over the Pacific Ocean in each hemisphere. This subdivision is due to the mutual influence of oceans and continents through the induced temperatures, and the configurations of the great currents of the general circulation depending upon them. The high-pressure belt is itself produced, in that latitude, by the downflow of air which has originally risen in the tropics; the segregation is accompanied by low-level "leakage currents" from the tropics to the temperate zones, which form the warm parts of cyclones and anticyclones. The cold streams from the polar zones meet these warm currents in mid-latitudes, and their interaction produces the local circulations of storms, under the force of gravitation acting on warm and cold masses in contact with each other.

For our special purpose in this connection, there has been much discussion regarding the existence of the "antitrade" winds blowing eastward in the upper levels, as above 12,000 meters. The observational data are themselves conflicting, but this points to a very important feature in the theory of the planetary circulation. Collecting some of the data for very high altitudes, 9,000 to 17,000 meters, we have the following typical exhibit:

TABLE 28

THE NUMBER OF WINDS FROM EIGHT COMPASS POINTS, IN DIFFERENT LATITUDES, AT 9,000 TO 17,000 METERS ALTITUDE

Station	N.	N.E.	E.	S.E.	S.	S.W.	W.	N.W.	To'l	Lat.
Lindenburg...........	18	17	14	4	1	16	2	9	81	52°
Ponta Delgada.........	6	16	8	4	1	10	12	9	66	37°.7
and Madeira........	...	...	...	...	...	...	...	...	...	32°.6
Teneriffe.............	3	0	2	2	3	22	1	5	38	28°.5
St. Vincent...........	0	2	13	4	5	2	0	0	26	17°
Victoria Nyanza.......	5	10	36	10	2	4	4	10	81	0°
Ascension.............	1	0	5	0	0	0	0	8	14	−8°

It is evident that there is an alternation of wind direction between N.E. and S.W. in all latitudes, but that the westward wind prevails over the equator, and to some extent predominates in all latitudes to 50° or 60°. This can only mean that in the isothermal region the temperatures increase from the equator to that latitude, as on Fig. 13, Case II, "High Temperatures in the Isothermal Region"; for all the wind directions blowing eastward, the opposite temperature gradient must occur, of fall from the equator toward the pole, as in Case I, "Low Temperatures in the Isothermal Region." We infer that in the general circulation there are heat maxima, or warm crests, which form near the equator, Case I, and move toward the pole to middle latitudes, Case II. The temperature gradients in the isothermal region are, therefore, very unsteady at any place, and there is a continuous mixture of the air currents, along with a vigorous radiation of heat from below, as will be further indicated. The computed data of Case II are here produced in Tables 29–42. While there are instances of "antitrade" winds at high elevations, the "trade" winds apparently penetrate to very high altitudes at other times. These observations were all made on the eastern side of the north Atlantic Tropic Ocean, and on the eastern edge of the high-pressure section of that region, where westward winds from N.E.- to S.W.-ward prevail as part of the forced circulation. On the western side of the ocean and western edge of this same section, it is probable that the "antitrade" wind will predominate much more vigorously, but this is again a localized effect of temperature and pressure distribution. An inspection of the Victoria Nyanza temperatures, Table 27, indicate a very pronounced fall of temperature on the levels from 16,000 to 18,000 meters, such as would produce a violent westward circulation, which does not seem to exist. It would be proper to confirm these valuable observations at other points over the equator, whenever practical. Similarly the temperatures of St. Louis relative to Europe would demand a violent westward wind at the high levels, which likewise does not regularly exist. These facts show how difficult it is to construct a satisfactory system of temperatures and velocities for the planetary circulation. There is great need for high-level balloon ascensions recording temperature, humidity, and wind velocity and direction, upon which to base the computations for the other terms in the problems of the atmosphere.

The Thermodynamic Tables of the Planetary Circulation and Radiation

After the preceding explanations regarding the observational data and formulas, the reader can easily study the results of the computations for Case II, in Tables 29–42, so that only special points of interest will be indicated.

Table 29. The *pressure* maximum is near latitude 30° at all elevations; the minimum near the pole is much lower than that at the equator.

Table 30. The *temperature* maximum is in the high-pressure maximum throughout the convectional region, and in the isothermal region it moves from near the equator toward the pole; there is a sharp drop in the temperature in passing from the convectional to the isothermal region; this boundary is located at 9,000–12,000 meters on the poleward side of 40° and it lies between 12,000 and 16,000 in the tropics; when the temperatures in the isothermal region are relatively cold the boundary is at high elevations, and when warm at low elevations respectively; over anticyclones the isothermal region is at high altitudes, and over cyclones at low latitudes; it is high in winter and low in summer; its elevation depends upon the temperature and gravitation conditions in the convection region and not on any inherent forces of its own; it is distinguished in its physical properties from the convectional by some properties which will be indicated under the topic of radiation.

Table 31. The *density* has nearly the same value on the same level of the tropics as a broad minimum, and it increases toward the pole, much more in the convectional than in the isothermal region.

Table 32. The *gas coefficient* and the dependent *specific heat* are variables, though assumed to be constant at the surface before radiation changes it with the elevation, but decreasing upward generally, much more near the pole than over the equator; there is an irregularity in passing to the isothermal region, accompanied by the change of temperature and velocity of circulation. The check $P = \rho T R$ is confirmed at every point.

TABLE 29

The Planetary Circulation and Radiation

The pressure P in the units of force ($M. K. S.$)

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	5795	5854	5964	6184	6356	6587	6945	6882	6637	6445
18000	6768	6838	6980	7224	7409	7709	8128	8038	7848	7683
17000	7922	8003	8148	8433	8672	9023	9550	9457	9273	9182
16000	9247	9367	9536	9843	10110	10519	11177	11127	10955	10829
15000	10823	10935	11136	11521	11833	12311	13133	13084	12915	12807
14000	12667	12797	13028	13448	13794	14408	15407	15374	15215	15108
13000	14825	14978	15248	15740	16144	16837	18041	18028	17869	17754
12000	17351	17530	17846	18422	18821	19674	21076	21069	20899	20782
11000	20308	20513	20887	21560	22028	22972	24555	24523	24345	24224
10000	23768	24013	24405	25135	25781	26808	28481	28403	28220	28098
9000	27818	28105	28596	29367	30099	31177	32860	32726	32542	32418
8000	32509	32838	33367	34213	35012	36068	37721	37534	37344	37222
7000	37859	38226	38772	39666	40517	41530	43113	42877	42678	42558
6000	43930	44344	44876	45781	46666	47620	49111	48812	48605	48495
5000	50811	51251	51742	52631	53516	54414	55772	55427	55219	55110
4000	58606	59038	59464	60297	61156	61967	63164	62791	62576	62491
3000	67437	67816	68143	68867	69678	70378	71355	70963	70772	70707
2000	77432	77703	77877	78455	79164	79722	80428	80046	79862	79838
1000	88730	88838	88836	89168	89704	90104	90462	90126	89958	89950
000	101521	101388	101135	101149	101414	101588	101548	101216	101042	101058

Formula: $\log P = \log P_0 + \frac{nk}{k-1}(\log T - \log T_0)$. . . . (182)

TABLE 30

The Temperature T in Absolute Degrees Centigrade

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	219.0	219.2	219.4	220.4	220.8	218.6	214.8	210.9	203.0	193.0
18000	218.6	218.8	219.0	220.0	220.6	218.4	214.6	210.6	204.0	196.0
17000	218.3	218.5	218.7	219.7	220.4	218.2	214.4	210.3	205.0	199.0
16000	218.0	218.2	218.4	219.4	220.2	218.0	214.2	210.0	206.0	202.0
15000	217.7	217.9	218.1	219.1	220.0	217.8	214.0	211.0	208.0	205.0
14000	217.4	217.6	217.8	218.8	219.8	218.0	215.0	213.0	210.0	209.0
13000	217.2	217.4	217.6	218.6	219.6	219.0	218.0	216.2	215.0	214.0
12000	217.0	217.2	217.4	218.4	219.4	220.0	221.0	222.0	221.0	220.0
11000	216.8	217.0	217.2	218.2	219.2	221.0	226.4	228.3	227.0	226.0
10000	216.6	216.8	217.0	218.0	219.0	222.1	234.3	236.3	235.5	234.5
9000	216.4	217.0	218.5	220.3	222.2	230.4	243.6	245.4	244.2	243.2
8000	221.8	222.1	224.5	227.3	230.0	238.6	251.6	253.2	252.5	251.5
7000	227.0	227.5	230.5	234.8	238.0	246.2	259.2	260.5	259.5	258.5
6000	232.8	233.0	237.0	241.8	245.7	253.0	265.8	266.8	265.5	264.8
5000	237.3	239.0	243.0	248.3	253.0	259.5	271.7	271.5	270.5	269.5
4000	241.5	244.2	248.3	254.2	259.0	265.8	277.3	276.5	275.5	274.0
3000	245.8	248.7	253.3	259.7	264.8	271.3	282.9	281.4	280.0	279.0
2000	249.0	253.0	257.7	264.6	270.6	276.6	288.4	286.2	285.0	284.0
1000	252.5	257.0	261.9	269.0	276.0	281.6	292.0	290.0	289.5	289.0
000	255.0	260.0	265.0	273.0	281.0	288.0	299.5	298.5	298.7	298.0

The observations made in balloon and kite ascensions.

Table 33. The adopted *wind velocity* indicates an eastward movement with the positive sign (+), westward with the negative sign (−), and alternating with the (∓) signs; it is quite likely that further observations will enable us to improve this mean table of velocities; these must ultimately be so adjusted to the air masses associated with them that the sum of the momenta of rotation about the earth's axis shall be equal to zero in order that the period of the earth's rotation may be constant, as indicated by astronomical observations, and this involves the corresponding pressure, temperature, density, and radiation of heat from point to point throughout the entire atmosphere.

Table 34. The *kinetic energy* of circulation from one level to another acts as a balance in the action of gravitation against the pressure and heat terms in the general equation. This action is very pronounced in passing from the convectional to the isothermal region; it is strong in some parts of cyclones and anticyclones, and in tornadoes, being due to rapid changes in the temperatures for short distances. When the pressure and heat terms are deficient, the kinetic energy makes it up by an increase in the velocity; when in excess, it balances the same by decreasing the velocity; the numerous horizontal currents in the atmosphere, as seen by the cloud motions in different directions, exhibit this process which is incessantly at work adjusting these delicate differences between pressure and radiation to the controlling force of gravitation; these adjust themselves mutually at every point in the atmosphere, and are not propagated at long range from one distant point to another.

Table 35. The *hydrostatic pressure per unit density* is computed from Tables 29, 31, for use in the general equation (196), which must always be satisfied. It should be carefully remembered that the density is to be computed by equation (176), and not by equation (175). The term decreases upward, and generally from the equator to the pole, though there is a small maximum in the convectional region just north of the high-pressure belt, in the levels 5,000 to 9,000 meters.

Table 36. The *free heat on which radiation depends* gives the change in the heat contents of the unit mass per 1,000

TABLE 31

THE PLANETARY CIRCULATION AND RADIATION

The density ρ in kilograms per cubic meter

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	0.1810	0.1788	0.1776	0.1769	0.1754	0.1756	0.1753	0.1737	0.1700	0.1670
18000	.2022	.1997	.1986	.1976	.1956	.1964	.1961	.1950	.1915	.1892
17000	.2261	.2233	.2217	.2206	.2188	.2197	.2199	.2189	.2156	.2140
16000	.2524	.2498	.2480	.2463	.2440	.2456	.2459	.2458	.2428	.2416
15000	.2823	.2788	.2768	.2754	.2729	.2740	.2758	.2758	.2730	.2722
14000	0.3157	0.3118	0.3096	0.3074	0.3048	0.3064	0.3090	0.3093	0.3066	0.3061
13000	.3531	.3487	.3463	.3437	.3403	.3423	.3457	.3464	.3438	.3433
12000	.3949	.3900	.3872	.3835	.3796	.3824	.3861	.3870	.3843	.3840
11000	.4417	.4362	.4331	.4300	.4245	.4270	.4304	.4311	.4284	.4282
10000	.4939	.4878	.4844	.4796	.4748	.4765	.4783	.4786	.4758	.4759
9000	0.5524	0.5456	0.5415	0.5357	0.5300	0.5306	0.5295	0.5293	0.5265	0.5269
8000	.6172	.6094	.6043	.5972	.5902	.5885	.5841	.5835	.5806	.5818
7000	.6878	.6790	.6724	.6634	.6548	.6505	.6424	.6414	.6385	.6394
6000	.7645	.7546	.7660	.7346	.7240	.7170	.7047	.7033	.7008	.7016
5000	.8479	.8364	.8255	.8112	.7981	.7884	.7714	.7698	.7669	.7684
4000	0.9384	0.9249	0.9114	0.8935	0.8775	0.8647	0.8428	0.8418	0.8382	0.8402
3000	1.0370	1.0207	1.0041	0.9821	0.9628	0.9466	0.9191	0.9177	0.9148	0.9173
2000	1.1440	1.1244	1.1071	1.0775	1.0543	1.0344	1.0008	0.9998	0.9969	1.0001
1000	1.2604	1.2367	1.2125	1.1802	1.1523	1.1284	1.0880	1.0878	1.0850	1.0886
000	1.3870	1.3586	1.3296	1.2908	1.2574	1.2289	1.1812	1.1814	1.1785	1 1825

Formula: $\log \rho = \log \rho_0 + \frac{n}{k-1} (\log T - \log T_0)$. . . . (183)

TABLE 32

THE GAS COEFFICIENT R, WHICH IS A VARIABLE

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	146.14	149.87	153.08	158.60	164.12	171.56	184.39	186.48	192.38	199.92
18000	153.12	156.51	160.45	166.18	171.70	179.71	193.14	195.72	200.88	207.12
17000	160.47	164.01	168.01	174.01	179.86	188.24	202.54	205.43	209.77	214.42
16000	168.03	171.88	176.06	182.21	188.17	196.94	217.10	215.62	219.05	221.92
15000	176.09	179.98	184.86	190.95	197.10	206.29	222.48	224.88	227.53	229.53
14000	184.53	188.61	193.20	199.95	206.21	215.68	231.91	233.39	236.28	236.14
13000	193.29	197.56	202.87	209.44	216.00	224.58	239.38	240.73	241.75	241.68
12000	202.46	206.94	211.93	219.37	225.99	233.85	246.98	245.24	246.08	245.98
11000	212.07	216.76	222.04	229.79	236.72	243.45	251.97	249.17	250.38	250.28
10000	222.14	227.05	232.58	246.01	247.95	253.29	254.13	250.64	251.86	251.77
9000	232.68	237.39	241.68	248.85	255.56	255.06	254.74	251.96	253.09	253.01
8000	237.48	242.61	245.94	252.06	257.92	256.88	256.66	254.07	254.69	254.62
7000	242.48	247.47	250.16	254.65	259.99	259.30	258.96	256.62	257.57	257.51
6000	247.35	252.22	253.79	257.74	262.33	262.51	262.21	260.62	261.39	261.05
5000	252.54	256.39	257.92	261.31	265.05	265.98	266.11	265.19	266.19	266.15
4000	258.59	261.39	262.76	265.47	269.08	269.62	270.28	269.94	270.97	271.46
3000	265.12	267.15	267.91	270.02	273.29	274.04	274.43	274.72	276.27	276.28
2000	271.82	273.15	273.71	275.18	277.48	278.65	278.66	279.74	281.07	281.11
1000	278.81	279.51	279.75	280.38	282.05	283.56	284.74	285.70	286.38	285.93
000	287.03	287.03	287.03	287.03	287.03	287.03	287.03	287.03	287.03	287.03

Formula: $\log R = \log R_0 + (n-1)(\log T - \log T_0)$. . . . (184)
Check $P = T \rho R$ at every point. . . . . (173)

meters, and the table indicates how far the atmosphere has departed from the adiabatic state. The term increases upward, and generally from the equator to the pole, but there is a minimum on entering the isothermal region in middle latitudes and a region of marked irregular progression in the values. Unfortunately there is no way to compute this term directly, as it depends upon the evaluation of the specific heat and the velocity of the circulation, by equation (199). It is therefore a great loss to science when an observatory measures the temperatures, but not the humidity and wind velocity, at different levels, because the entire subject of thermodynamic meteorology is thereby excluded from further discussions. The magnitude of the term $(Q_1 - Q_0)$ can be seen in Table 36 to be very great in the upper levels, and that in comparison with it the kinetic energy of the circulation, as in Table 34, is very small. Hence, all those theories of atmospheric circulation which depend upon gravitation, pressure, and circulation alone, and omit heat changes through radiation, either absorption or emission, have no permanent value. The difficulty of determining the heat term has, no doubt, been the cause of this defect which prevails in meteorological literature, but it is none the less a fatal defect in this branch of science. Table 43, p. 131, gives the second differences of the heat contents, and it is the rate of change of the heat per 1,000 meters, or the radiation rate, which prevails on the average in all parts of the atmosphere, and it is fundamental to all studies of "solar constant" and solar insolation. This table will be discussed more fully under the subject of Bolometry and Pyrheliometry, and at this place only its leading features will be noted. Taking the hemisphere as a whole, the mean rate of radiation is − 157.1 in the convectional region, and − 283.1 in the isothermal region, almost twice as great in the latter. This changes greatly in latitude, as can be seen in the means for the isothermal region (I), and the convectional region (C), respectively. The means on the table are taken outside the layers of transition, and these give nearly a constant value in C from the equator to latitude 60°, but an increasing value to the pole; in the I region there is a minimum in the tropics, maximum in the middle latitudes,

TABLE 33

THE PLANETARY CIRCULATION AND RADIATION

The velocity q in meters per second

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	0.0	3.0	4.0	4.0	5.0	∓5.0	∓6.0	∓7.0	∓ 9.0	−12.0
18000	0.0	3.0	4.0	5.0	6.0	6.0	∓7.0	∓8.0	∓10.0	−12.0
17000	0.0	3.0	4.0	6.0	7.0	7.0	∓8.0	∓9.0	∓10.0	−12.0
16000	0.0	4.0	5.0	7.0	8.0	9.0	∓6.0	∓6.0	∓10.0	−11.0
15000	0.0	5.0	6.0	8.0	9.0	10.0	∓4.0	∓4.0	∓ 7.0	−10.0
14000	0.0	6.0	7.0	9.0	10.0	11.0	∓2.0	∓2.0	∓ 4.0	− 9.0
13000	0.0	6.0	8.0	10.0	11.0	12.0	4.0	5.0	∓ 2.0	− 8.0
12000	0.0	7.0	9.0	11.0	12.0	13.0	9.0	5.0	− 3.0	− 7.0
11000	0.0	7.0	11.0	13.0	15.0	16.0	12.0	6.0	− 4.0	− 5.0
10000	0.0	8.0	15.0	17.0	20.0	20.0	15.0	6.0	− 5.0	− 3.0
9000	0.0	5.6	25.2	31.6	35.0	34.0	21.7	0.8	− 3.8	− 2.0
8000	0.0	4.8	22.8	28.9	31.5	29.0	16.0	−2.6	− 5.0	− 2.5
7000	0.0	4.0	19.6	25.8	26.9	22.0	9.0	−4.9	− 6.2	− 2.6
6000	0.0	3.2	15.9	21.4	21.8	15.4	2.4	−6.8	− 7.0	− 2.7
5000	0.0	2.0	12.4	17.6	17.2	11.4	2.6	−8.6	− 7.4	− 2.8
4000	0.0	0.8	9.0	13.8	13.6	9.2	6.6	−9.4	− 7.6	− 2.8
3000	0.0	0.4	6.6	11.0	11.0	7.4	7.3	−9.4	− 7.2	− 2.8
2000	0.0	1.6	5.0	9.4	9.4	6.4	6.1	−8.4	− 6.2	− 2.4
1000	0.0	2.4	3.8	8.0	8.0	5.6	4.1	−6.7	− 4.6	− 1.4
000	0.0	3.4	3.4	7.4	7.1	5.1	2.0	−4.6	− 2.6	− 0.4

From the observations made in balloon and kite ascensions.

TABLE 34

THE KINETIC ENERGY OF CIRCULATION $-\frac{1}{2}(q^2_1 - q^2_0)$

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	0.0	0.0	0.0	+ 4.5	+ 5.5	+ 5.5	+ 6.5	+ 7.5	+ 9.5	0.0
18000	0.0	0.0	0.0	+ 5.5	+ 6.5	+ 6.5	+ 7.5	+ 8.5	0.0	0.0
17000	0.0	+ 3.5	+ 8.5	+ 6.5	+ 7.5	+ 16.0	− 14.0	−22.5	0.0	−11.5
16000	0.0	+ 4.5	+ 5.5	+ 7.5	+ 8.5	+ 9.5	− 10.0	−10.0	−25.5	−10.5
15000	0.0	+ 5.5	+ 6.5	+ 8.5	+ 9.5	+ 10.5	− 6.0	− 6.0	−16.5	− 9.5
14000	0.0	0.0	+ 7.5	+ 9.5	+ 22.0	+ 11.5	+ 6.0	+10.5	− 6.0	− 8.5
13000	0.0	+ 6.5	+ 8.5	+ 10.5	+ 11.5	+ 12.5	+ 32.5	+ 5.0	+ 2.5	− 7.1
12000	0.0	0.0	+20.0	+ 24.0	+ 40.5	+ 43.5	+ 31.5	+ 5.5	+ 3.5	−12.5
11000	0.0	+ 7.5	+52.0	+ 60.0	+ 87.5	+ 72.0	+ 40.5	− 0.0	+ 4.5	− 8.0
10000	0.0	+16.3	−54.9	+354.8	+412.5	+378.0	+123.0	−17.7	− 5.3	− 2.5
9000	0.0	− 4.2	−57.6	−81.3	−116.4	−157.5	−107.4	+ 3.1	+ 5.8	+ 1.1
8000	0.0	− 3.5	−67.8	− 84.8	−134.3	−178.5	− 87.5	+ 8.6	+ 6.7	+ 0.3
7000	0.0	− 2.9	−65.7	−103.8	−124.2	−123.4	− 37.6	+11.1	+ 5.3	+ 0.2
6000	0.0	− 3.1	−49.5	− 74.1	− 89.7	− 53.6	+ 0.5	+13.9	+ 2.9	+ 0.2
5000	0.0	− 1.7	−36.4	− 59.7	− 55.4	− 22.7	+ 18.4	+ 7.2	+ 1.5	0.0
4000	0.0	− 0.2	−18.7	− 34.7	− 32.0	− 14.9	+ 4.8	0.0	− 3.0	0.0
3000	0.0	+ 1.2	− 9.3	− 16.3	− 16.3	− 6.9	− 8.0	− 8.9	− 6.7	− 1.0
2000	0.0	+ 1.6	− 5.3	− 12.2	− 12.2	− 4.8	− 10.2	−12.8	− 8.6	− 1.9
1000	0.0	+ 2.9	− 1.4	− 4.6	− 6.8	− 2.7	− 6.4	−11.8	− 7.2	− 0.9
000	0.0	......	......	....	....	....	....	......	......	......

To obtain the kinetic energy these numbers must be multiplied by the appropriate values of ρ_{18}.

and minimum in the polar zone. The broad fact is clear that *about twice as much heat radiates through the isothermal region as through the convection region.* At the strata of transition the radiation is in some latitudes positive, as at the 10,000-meter level from 30° to 60° latitude. These *special sources of heat* will be further considered in discussing the value of the "solar-constant" of radiation. The importance of such data in radiation problems is very great, because it seems to explain several processes in the atmosphere that have been very obscure.

Table 37. The *entropy* increases upward and from the equator to the pole, though the same variation occurs in the transition to the isothermal region as was noted for the heat contents.

Table 38. The *work against external forces,* such as expansion when a mass of air is raised from one level to another in the circulation, is to be computed through a new value of R', which differs from the R of (184) in consequence of the elimination of the velocity of the circulation, as explained in (199), (333), on Table 19. It is the circulation term which continually interferes with a perfect check to this series of equations, and on that account, therefore, must be most carefully observed.

Table 39. The *inner energy* is the heat contents remaining after the work has been done externally, and it increases upward and from the equator to the pole, with the same variation between the convectional and the isothermal regions.

Table 40. The *radiation function* is the rate of change of the inner energy with the change of the volume, and it may be computed by several formulas. The K_{10} differs from P_{10} by the rate of change of the heat contents with the change in volume, or it may be seen to depend upon the variation of the gas coefficient R. The value of K_{10} generally decreases with the altitude, and there is a small maximum in the high-pressure belt of the convectional region, but this shifts toward the pole in the isothermal region. The immediate problem is to determine the relation to the temperature through the approximate Stefan formula, and this has been done in the same way as that followed

TABLE 35

The Planetary Circulation and Radiation

The Hydrostatic Pressure per Unit Density $-\dfrac{P_1 - P_0}{\rho_{10}}$

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	5078.2	5198.1	5401.4	5552.5	5676.6	6032.3	6370.4	6540.2	6697.9	6651.2
18000	5387.5	5508.6	5556.6	5782.0	6095.6	6314.3	6836.6	6855.0	6998.9	7187.5
17000	5537.1	5765.0	5909.0	6038.5	6214.4	6428.8	6985.8	7185.8	7338.7	7449.5
16000	5898.9	5932.7	6082.3	6431.6	6665.4	6897.7	7497.2	7503.8	7600.0	7699.7
15000	6167.3	6305.5	6466.7	6532.2	6794.3	7226.0	7777.0	7826.5	7936.5	7956.5
14000	6453.3	6603.1	6766.3	6962.3	7291.3	7490.0	8045.2	8094.0	8161.2	8149.2
13000	6754.0	6908.4	7082.8	7366.2	7436.2	7828.3	8294.7	8293.0	8321.8	8325.7
12000	7069.0	7220.8	7413.5	7704.4	7975.7	8149.3	8520.6	8445.0	8479.5	8475.8
11000	7396.3	7575.8	7668.0	7860.7	8345.6	8490.4	8640.0	8531.2	8571.2	8569.0
10000	7740.8	7919.5	8169.8	8335.6	8594.6	8675.4	8690.2	8577.4	8623.2	8615.8
9000	8021.6	8195.8	8327.8	8555.8	8771.6	8740.2	8730.4	8641.2	8674.0	8669.8
8000	8199.2	8364.0	8466.6	8643.6	8843.4	8816.8	8799.4	8724.6	8750.0	8742.0
7000	8359.8	8595.0	8606.8	8748.2	8919.2	8906.2	8897.0	8826.6	8854.0	8854.6
6000	8535.2	8702.6	8737.6	8862.8	9001.4	9026.2	9025.6	8980.5	9015.7	9000.0
5000	8727.2	8842.8	8892.2	8993.6	9119.0	9137.4	9158.6	9141.0	9166.4	9177.0
4000	8940.8	9023.6	9061.5	9138.4	9261.0	9287.8	9297.4	9291.6	9350.8	9349.0
3000	9165.4	9217.6	9234.4	9310.4	9405.0	9433.6	9451.0	9473.2	9509.4	9524.5
2000	9397.8	9432.0	9461.2	9490.8	9553.0	9600.4	9607.2	9657.0	9698.4	9682.0
1000	9663.0	9671.6	9676.5	9697.5	9719.4	9743.6	9770.8	9776.8	9793.3	9781.4
000	......	......	......	......	......	......	......	......	......	......

Computed from Tables 29, 31.

TABLE 36

The Heat Which Gives Rise to Radiation $-(Q_1 - Q_0)$

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	4694.2	4581.1	4451.2	4253.8	4064.2	3800.2	3350.6	3270.0	3079.6	2853.1
18000	4449.3	4331.0	4195.5	3982.7	3794.3	3514.3	2990.1	2944.8	2791.4	2605.0
17000	4204.4	4064.9	3920.2	3707.8	3512.0	3210.4	2736.1	2677.4	2481.0	2363.8
16000	3927.8	3793.2	3643.8	3424.3	3216.4	2908.5	2391.6	2291.5	2203.3	2104.8
15000	3645.9	3504.4	3350.2	3140.1	2907.2	2587.5	2050.2	1934.7	1900.1	1861.0
14000	3352.2	3131.0	3041.4	2803.6	2571.7	2273.9	1749.7	1696.7	1646.3	1653.7
13000	3045.8	2811.2	2719.8	2470.7	2244.6	1962.7	1465.6	1499.7	1470.5	1483.8
12000	2724.1	2568.5	2372.2	2109.6	1866.1	1609.4	1233.1	1355.2	1322.5	1341.5
11000	2388.8	2217.5	1988.4	1714.0	1439.6	1248.7	1120.4	1268.5	1222.6	1238.1
10000	2036.9	1888.9	1759.7	1093.8	792.7	744.4	990.6	1238.5	1185.9	1185.9
9000	1774.7	1611.1	1534.0	1331.4	1151.3	1218.7	1177.8	1159.1	1126.8	1133.6
8000	1607.5	1438.2	1399.5	1235.4	1093.6	1167.3	1085.8	1074.0	1049.0	1057.6
7000	1439.1	1273.3	1263.3	1157.2	1008.1	1016.1	941.2	959.0	936.0	948.0
6000	1267.2	1121.2	1114.6	1013.9	887.3	832.2	781.0	810.4	791.2	800.4
5000	1075.2	958.1	948.3	867.6	736.2	697.8	625.3	658.0	628.9	622.9
4000	860.2	777.7	760.0	693.8	573.4	534.2	496.8	501.1	461.1	449.7
3000	634.2	575.6	563.7	559.4	414.2	372.3	366.3	342.5	292.4	285.9
2000	400.3	364.2	357.3	319.7	260.5	202.4	192.4	160.1	121.7	122.1
1000	140.4	125.2	125.8	109.7	91.9	62.1	45.5	34.6	18.5	24.4
000	....	....	....	....	....	....	....	....	....	....

Formula: $-(Q_1 - Q_0) = -(Cp_a - Cp_{10})(T_a - T_0) + \frac{1}{2}(q_1^2 - q_0^2)$ . . . (199).

Check $q(z_1 - z_0) = -\dfrac{P_1 - P_0}{\rho_{10}} - \frac{1}{2}(q_1^2 - q_0^2) - (Q_1 - Q_0)$ . . . . (196).

Comparing the numerical values of these terms, it is seen how impossible it is to solve any problems in dynamic meteorology without the term $-(Q_1 - Q_0)$.

on Table 20, the intermediate steps of computing log C and A for the ratios K_1/K_0 and T_1/T_0 being here omitted.

Table 41. *The log c* in the formula diminishes slowly from the surface upward, it increases from the equator to the pole in the convectional region, but decreases in the isothermal region, the result being a nearly uniform value on the high levels of the isothermal region. The minus sign affects only the characteristic of the logarithm, and for a mean value $\log c = -5.500$, $c = 3.162 \times 10^{-5}$. The Kurlbaum coefficient in the Stefan formula for a perfect radiator is taken at 7.68×10^{-11} (C. G. min. C°) $= 5.32 \times 10^{-6}$ joules per square meter per sec., so that the air radiates at six times the rate of a perfect radiator in the ether, as when the sun transmits its radiant energy to the earth. The quantity c relates to a summation of the successive radiations from one air volume to another throughout a layer 1,000 m. deep. The number of repeated transfers in this depth can only be approximately inferred from the knowledge that the efficient radiating layer of the atmosphere, or that for which any further increase of depth does not add to the radiant intensity, amounts to several meters. The latter quantity is considerably smaller than the superficial radiation from a black solid.

Table 42. The *exponent a* diminishes from the surface upward; it increases from the equator to the pole in the convectional region, but diminishes in the isothermal region. Its mean value is 3.82 instead of 4.00, as in the Stefan formula, at the surface. Comparing the data we have for the

Perfect Radiator, $K_{10} = 5.32 \times 10^{-6}\ T_{10}^{4.00}$ joule/sq. m. sec.

(K. M. S. system) and for the

Atmosphere, $K_{10} = 3.162 \times 10^{-5}\ T_{10}^{3.82}$ per 1,000 m.

The consequences of these data in theories of atmospheric radiation will require much careful investigation in several directions.

TABLE 37

THE PLANETARY CIRCULATION AND RADIATION

The Entropy $(S_1 - S_0)$

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	−21.454	−20.919	−20.307	−19.818	−18.415	−17.392	−15.606	−15.516	−15.133	−14.669
18000	−20.363	−19.799	−19.166	−18.111	−17.208	−16.099	−13.938	−13.993	−13.650	−13.190
17000	−19.269	−18.612	−17.933	−16.885	−15.942	−14.720	−12.768	−12.740	−12.073	−11.790
16000	−18.026	−17.392	−16.692	−15.615	−14.613	−13.348	−11.171	−10.884	−10.644	−10.343
15000	−16.755	−16.090	−15.363	−14.338	−13.221	−11.875	− 9.558	− 9.126	− 9.091	− 8.990
14000	−15.427	−13.986	−13.971	−12.819	−11.705	−10.407	− 8.082	− 7.906	− 7.748	− 7.819
13000	−14.029	−12.937	−12.505	−11.308	−10.226	− 8.942	− 6.677	− 6.845	− 6.745	− 6.838
12000	−12.559	−11.881	−10.917	− 9.664	− 8.509	− 7.299	− 5.512	− 6.019	− 5.904	− 6.019
11000	−11.024	−10.224	− 9.159	− 7.859	− 6.570	− 5.685	− 4.864	− 5.330	− 5.287	− 5.377
10000	− 9.408	− 8.510	− 8.079	− 4.990	− 3.593	− 3.291	− 4.146	− 5.081	− 4.944	− 4.965
9000	− 8.100	− 7.337	− 6.926	− 5.949	− 5.092	− 5.197	− 4.757	− 4.649	− 4.537	− 4.583
8000	− 7.164	− 6.398	− 6.152	− 5.348	− 4.672	− 4.816	− 4.251	− 4.182	− 4.098	− 4.147
7000	− 6.268	− 5.531	− 5.403	− 4.836	− 4.169	− 4.071	− 3.728	− 3.643	− 3.566	− 3.623
6000	− 5.397	− 4.751	− 4.644	− 4.138	− 3.558	− 3.248	− 2.905	− 3.014	− 2.952	− 2.996
5000	− 4.491	− 3.966	− 3.861	− 3.454	− 2.876	− 2.589	− 2.278	− 2.402	− 2.304	− 2.992
4000	− 3.534	− 3.156	− 3.030	− 2.700	− 2.181	− 1.989	− 1.774	− 1.796	− 1.660	− 1.626
3000	− 2.566	− 2.295	− 2.206	− 2.134	− 1.547	− 1.359	− 1.283	− 1.207	− 1.035	− 1.016
2000	− 1.596	− 1.428	− 1.375	− 1.198	− 0.953	− 0.725	− 0.663	− 0.609	− 0.424	− 0.426
1000	− 0.553	− 0.484	− 0.478	− 0.405	− 0.330	− 0.218	− 0.154	− 0.118	− 0.063	− 0.083
000	....	....	....	....	....	....	....	....	....	....

Formula: $(S_1 - S_0) = \frac{Q_1 - Q_0}{T_{10}}$. . . (331).

TABLE 38 THE EXTERNAL WORK $(W_1 - W_0)$

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	3601.4	3688.7	3854.4	3948.5	4017.9	4256.9	4344.3	4652.0	4754.7	4942.6
18000	3849.6	3926.9	3985.8	4099.8	4358.9	4496.7	4867.6	4872.0	4972.4	5107.2
17000	3918.8	4109.2	4208.6	4276.8	4396.1	4523.4	4943.4	5126.4	5222.6	5299.5
16000	4195.8	4195.7	4302.1	4588.0	4761.7	4905.1	5355.2	5333.0	5403.6	5474.9
15000	4387.8	4485.0	4601.7	4606.5	4801.8	5140.7	5536.4	5552.7	5652.6	5661.2
14000	4588.9	4645.9	4812.1	4939.4	5201.4	5314.0	5717.8	5751.3	5804.0	5794.1
13000	4801.0	4887.7	5035.7	5247.1	5251.8	5562.5	5885.3	5893.3	5913.8	5921.5
12000	5023.1	5129.9	5266.0	5581.0	5681.9	5781.4	6044.0	6003.6	6028.7	6030.5
11000	5253.5	5383.6	5409.6	5523.0	5928.7	6018.3	6130.8	6065.7	6091.6	6093.9
10000	5496.4	6532.3	5845.3	5818.7	6019.3	6056.5	6143.5	6102.3	6133.0	6125.6
9000	5701.4	5828.4	5938.1	6107.6	6271.4	6259.4	6237.8	6143.2	6166.6	6164.4
8000	5830.8	5945.7	6038.0	6167.7	6326.5	6321.2	6280.2	6202.0	6220.2	6214.7
7000	5942.7	6130.0	6138.9	6249.8	6377.6	6366.8	6336.1	6271.0	6291.2	6295.6
6000	6068.4	6193.7	6226.8	6322.9	6424.9	6433.7	6418.4	6381.8	6411.4	6398.4
5000	6205.0	6286.7	6333.3	6411.4	6498.8	6501.0	6506.4	6498.2	6515.2	6524.1
4000	6356.4	6415.4	6448.2	6506.0	6593.8	6609.2	6608.1	6603.5	6651.2	6646.0
3000	6515.8	6551.0	6564.4	6639.2	6722.8	6708.4	6724.1	6739.3	6761.0	6774.3
2000	6680.6	6704.3	6731.6	6750.3	6795.4	6826.1	6830.0	6874.9	6900.7	6884.4
1000	6870.7	6874.9	6880.0	6896.4	6911.3	6928.7	6951.1	6953.9	6965.8	6955.6
000	......	......	......	......	......	......	......	......	......	......

Formula: $(W_1 - W_0) = R'_{10}(T_a - T_0) - \frac{P_1 - P_0}{\rho_{10}}$ . . . . (333).

$R'_{10} = \frac{k-1}{k}\left(Cp_a - \frac{Q_1 - Q_0}{T_a - T_0}\right) = \frac{k-1}{k} Cp'_{10}$ . . . (349) and (350).

TABLE 39. THE PLANETARY CIRCULATION AND RADIATION

The Inner Energy $(U_1 - U_0)$

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	−8295.6	−8269.8	−8305.6	−8202.3	−8082.1	−8057.1	−7694.9	−7922.0	−7834.3	−7795.7
18000	−8298.9	−8257.9	−8131.3	−8082.5	−8153.2	−8011.0	−7857.7	−7817.7	−7763.8	−7712.2
17000	−8123.2	−8174.1	−8128.8	−7984.6	−7908.1	−7733.8	−7679.5	−7803.6	−7703.6	−7663.3
16000	−8123.6	−7988.9	−7945.9	−8012.3	−7978.4	−7813.6	−7746.8	−7624.5	−7606.9	−7579.7
15000	−8033.7	−7989.4	−7951.9	−7746.6	−7709.9	−7728.2	−7586.6	−7487.4	−7552.7	−7522.2
14000	−7941.1	−7776.9	−7853.5	−7743.0	−7773.1	−7587.9	−7467.5	−7448.0	−7450.5	−7447.8
13000	−7846.8	−7698.9	−7755.5	−7717.8	−7496.4	−7525.2	−7350.9	−7393.0	−7384.3	−7405.0
12000	−7747.2	−7698.4	−7638.2	−7690.6	−7548.0	−7390.8	−7277.1	−7358.8	−7341.2	−7372.0
11000	−7642.3	−7601.1	−7398.0	−7237.0	−7368.3	−7267.0	−7251.2	−7334.2	−7314.2	−7332.0
10000	−7533.3	−7521.2	−7605.0	−6912.5	−6812.0	−6800.9	−7134.1	−7340.8	−7318.9	−7311.5
9000	−7476.1	−7439.5	−7472.1	−7439.0	−7422.7	−7478.1	−7415.6	−7302.3	−7293.4	−7298.0
8000	−7438.3	−7383.9	−7437.5	−7403.1	−7420.1	−7488.5	−7366.0	−7276.0	−7269.2	−7272.3
7000	−7381.8	−7403.3	−7402.2	−7407.0	−7385.7	−7382.9	−7277.3	−7230.5	−7227.6	−7243.6
6000	−7335.6	−7314.9	−7341.4	−7336.8	−7312.2	−7265.9	−7199.4	−7192.2	−7202.6	−7198.8
5000	−7280.2	−7244.8	−7281.6	−7279.0	−7235.0	−7180.8	−7131.7	−7156.2	−7144.1	−7147.0
4000	−7216.6	−7193.1	−7208.2	−7199.8	−7167.2	−7143.4	−7104.9	−7104.6	−7112.3	−7095.8
3000	−7150.0	−7126.6	−7128.1	−7198.6	−7137.0	−7080.7	−7090.4	−7081.8	−7053.4	−7060.2
2000	−7080.9	−7068.5	−7088.9	−7070.0	−7055.9	−7028.5	−7022.4	−7035.0	−7022.4	−7006.5
1000	−7011.1	−7000.1	−7005.8	−7006.1	−7003.2	−7010.8	−6996.6	−6988.5	−6984.3	−6980.0
000	....	....	....	....	....	....	....	....	....	....

Formula: $(U_1 - U_0) = (Cp_a - R_{10})(T_a - T_0) = (Q_1 - Q_0) - (W_1 - W_0)$. . . . (335).

TABLE 40. THE POTENTIAL RADIATION ENERGY K_{10} PER 1,000 METERS

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	−14362	−14136	−13938	−13874	−13719	−13373	−12744	−12606	−11874	−11086
18000	−15850	−15578	−15503	−15331	−15068	−14866	−14222	−13963	−13299	−12620
17000	−17632	−17038	−17031	−16909	−16744	−16441	−15989	−15648	−14855	−14367
16000	−19375	−19151	−18910	−18638	−18387	−18087	−17586	−17215	−16696	−16276
15000	−21424	−21046	−20801	−20483	−20356	−20000	−19493	−19057	−18769	−18473
14000	−23691	−22906	−22970	−22483	−22356	−22180	−21727	−21507	−21148	−21021
13000	−26174	−25366	−25369	−25066	−24684	−24568	−24276	−24415	−24077	−24004
12000	−28907	−28365	−27948	−27945	−26154	−27902	−27286	−27821	−27423	−27416
11000	−31882	−31332	−30270	−30104	−30752	−29832	−31015	−31874	−31431	−31360
10000	−35153	−34660	−34901	−31666	−31020	−31735	−35528	−36650	−36140	−35964
9000	−39369	−38726	−38938	−38724	−38599	−40445	−42015	−41632	−41206	−41092
8000	−44702	−43952	−44377	−44277	−44405	−46198	−47431	−47002	−46597	−46528
7000	−50594	−51058	−50423	−50698	−50587	−51810	−52850	−52662	−52261	−52225
6000	−57042	−56446	−56866	−57096	−57038	−57574	−58674	−58568	−58133	−58101
5000	−63973	−63329	−63875	−64076	−63801	−64113	−64951	−64879	−64420	−64213
4000	−71312	−70868	−71085	−71285	−70963	−71363	−72057	−71692	−71123	−70887
3000	−79180	−78922	−79026	−79897	−79214	−79024	−79848	−79216	−78372	−78270
2000	−87742	−87482	−87624	−87500	−87434	−87200	−87670	−86958	−86162	−86180
1000	−96840	−96420	−96408	−96502	−96594	−96696	−96505	−95995	−95542	−95618
000	....	....	....	....	....	....	....	....	....	....

Formula: $K_{10} = \frac{U_1 - U_0}{v_1 - v_0} = \frac{Q_1 - Q_0}{v_1 - v_0} - P_{10} = -P_{10}\frac{Ra}{R_{10}}$. . . (339).

$\frac{K_1}{K_0} = \left(\frac{T_1}{T_0}\right)^A$ $K_{10} = c\,T_{10}^{A}$. . . (340).

TABLE 41

THE PLANETARY CIRCULATION AND RADIATION

The coefficient log c in $\log K_{10} = \log c + a \log T_{10}$ . . . (344)

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	−5.383	−5.416	−5.386	−5.380	−5.381	−5.378	−5.361	−5.387	−5.419	−5.433
18000	−5.404	−5.434	−5.408	−5.406	−5.404	−5.404	−5.387	−5.409	−5.426	−5.449
17000	−5.423	−5.445	−5.431	−5.426	−5.418	−5.424	−5.418	−5.439	−5.454	−5.465
16000	−5.436	−5.474	−5.454	−5.449	−5.444	−5.447	−5.435	−5.454	−5.469	−5.480
15000	−5.459	−5.486	−5.475	−5.468	−5.460	−5.466	−5.477	−5.463	−5.481	−5.491
14000	−5.481	−5.503	−5.496	−5.489	−5.484	−5.489	−5.490	−5.496	−5.504	−5.500
13000	−5.503	−5.522	−5.520	−5.513	−5.494	−5.504	−5.503	−5.518	−5.521	−5.510
12000	−5.526	−5.526	−5.542	−5.538	−5.510	−5.523	−5.507	−5.505	−5.507	−5.515
11000	−5.559	−5.571	−5.560	−5.544	−5.571	−5.540	−5.514	−5.510	−5.518	−5.518
10000	−5.578	−5.580	−5.589	−5.560	−5.547	−5.531	−5.512	−5.511	−5.513	−5.519
9000	−5.606	−5.601	−5.595	−5.589	−5.569	−5.552	−5.526	−5.511	−5.513	−5.519
8000	−5.614	−5.611	−5.596	−5.603	−5.563	−5.555	−5.524	−5.514	−5.517	−5.524
7000	−5.622	−5.618	−5.608	−5.608	−5.576	−5.556	−5.557	−5.524	−5.526	−5.531
6000	−5.628	−5.626	−5.610	−5.607	−5.577	−5.558	−5.536	−5.532	−5.537	−5.543
5000	−5.637	−5.627	−5.615	−5.615	−5.581	−5.564	−5.547	−5.531	−5.527	−5.536
4000	−5.638	−5.635	−5.622	−5.600	−5.589	−5.574	−5.534	−5.537	−5.542	−5.548
3000	−5.656	−5.646	−5.628	−5.616	−5.588	−5.561	−5.546	−5.554	−5.555	−5.561
2000	−5.664	−5.652	−5.634	−5.614	−5.597	−5.582	−5.561	−5.588	−5.546	−5.549
1000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....

The minus sign affects only the characteristic $\log c = -5.664$.
$c = 4.61 \times 10^{-5}$

TABLE 42

THE EXPONENT a IN $K_{10} = c\, T_{10}^{\,a}$ . . . (344)

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	3.740	3.755	3.741	3.739	3.739	3.738	3.730	3.742	3.757	3.763
18000	3.750	3.763	3.752	3.750	3.750	3.749	3.742	3.752	3.764	3.770
17000	3.758	3.768	3.762	3.760	3.756	3.759	3.756	3.766	3.772	3.777
16000	3.764	3.781	3.772	3.770	3.768	3.769	3.764	3.772	3.779	3.783
15000	3.775	3.786	3.782	3.778	3.775	3.778	3.782	3.776	3.784	3.789
14000	3.784	3.794	3.791	3.788	3.785	3.788	3.788	3.791	3.796	3.793
13000	3.794	3.803	3.802	3.798	3.790	3.794	3.794	3.801	3.802	3.797
12000	3.804	3.804	3.812	3.810	3.797	3.803	3.796	3.795	3.796	3.799
11000	3.820	3.825	3.820	3.813	3.825	3.811	3.799	3.797	3.799	3.801
10000	3.828	3.829	3.833	3.820	3.814	3.807	3.798	3.798	3.799	3.801
9000	3.841	3.838	3.835	3.833	3.824	3.816	3.804	3.798	3.799	3.802
8000	3.844	3.843	3.836	3.833	3.822	3.818	3.804	3.799	3.800	3.804
7000	3.848	3.846	3.841	3.833	3.823	3.818	3.818	3.804	3.804	3.807
6000	3.850	3.850	3.842	3.831	3.828	3.819	3.809	3.807	3.810	3.812
5000	3.854	3.850	3.844	3.837	3.829	3.822	3.814	3.807	3.805	3.809
4000	3.855	3.854	3.848	3.838	3.833	3.826	3.808	3.810	3.812	3.814
3000	3.863	3.858	3.850	3.845	3.832	3.820	3.814	3.817	3.818	3.820
2000	3.867	3.862	3.853	3.844	3.836	3.830	3.820	3.832	3.814	3.815
1000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....

The exponent $a = 3.50$ in dry air; $a = 4.00$ in a perfect radiator.

TABLE 43

The Radiation of Heat per 1,000 Meters

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°	Means
Case I.—Low Temperatures in the Isothermal Region											
19	−282.6	−298.6	−306.5	−307.4	−306.3	−324.3	−341.4	−329.8	−331.1	−319.7	−325.8
18	−296.2	−297.7	−315.2	−320.5	−324.2	−348.7	−329.8	−349.4	−345.3	−331.6	−325.8
17	−309.5	−316.2	−319.9	−339.5	−327.3	−317.6	−362.8	−337.2	−341.7	−343.6	−331.4
16	−322.2	−325.2	−329.4	−336.5	−337.4	−356.2	−336.9	−285.0	−306.8	−312.2	−324.7
15	−337.5	−344.8	−344.4	−336.6	−357.5	−344.2	−265.5	−202.6	−231.7	−235.3	−300.1
14	−345.8	−351.6	−358.7	−361.2	−326.1	−315.0	−151.5	−162.4	−167.9	−169.8	−271.0
13	−269.6	−275.2	−253.0	−273.6	−301.6	−200.5	− 92.9	−142.3	−153.3	−148.7	−211.0
12	−196.3	−194.3	−179.1	−175.5	−176.4	−109.9	− 49.0	− 98.6	−101.8	− 99.6	−138.1
11	−189.6	−191.1	−144.5	−103.3	− 70.8	+ 25.9	− 23.9	− 65.7	− 71.1	− 46.4	− 86.0
10	−173.3	−181.9	−123.8	− 83.9	− 57.9	− 25.1	− 23.0	− 71.5	− 47.9	− 49.1	− 83.7

CASE II.—HIGH TEMPERATURES IN THE ISOTHERMAL REGION

19	−244.9	−250.1	−255.7	−271.1	−269.9	−285.9	−360.5	−325.2	−288.2	−248.1	−280.0
18	−244.9	−266.1	−275.3	−274.9	−282.3	−303.9	−354.0	−367.4	−310.4	−241.2	−272.0
17	−276.6	−271.7	−276.4	−283.5	−295.6	−301.9	−344.5	−385.9	−277.7	−259.0	−297.3
16	−281.9	−288.8	−293.6	−284.2	−309.2	−321.0	−341.4	−356.8	−303.2	−243.8	−302.4
15	−293.7	−273.4	−308.8	−336.5	−335.5	−313.6	−270.5	−238.0	−253.6	**−207.3**	−296.1
14	−306.4	−320.8	−321.6	−332.9	−327.1	−311.2	−284.1	**−197.0**	**−176.0**	**−169.9**	−274.6
13	−321.7	−342.7	−347.6	−361.1	−378.5	−353.3	−232.5	**−144.5**	**−148.0**	**−142.3**	−267.2
12	−335.3	−351.0	−383.8	−395.6	−426.5	−360.7	**−112.7**	**− 86.7**	**− 99.9**	**−103.4**	−265.5
11	−351.9	−328.6	−228.7	**−620.2**	**−646.9**	**−504.3**	**−129.8**	**− 30.0**	**− 36.7**	**− 52.2**	−293.0
10	−262.2	−277.8	−235.7	**+237.6**	**+358.6**	**+474.3**	**+187.2**	**− 79.4**	**− 59.1**	**− 52.3**	−169.8
9	−167.2	−162.9	−144.5	**− 96.0**	**− 57.7**	**− 51.4**	**− 92.0**	**− 85.1**	**− 77.8**	**− 76.0**	−101.1
8	−168.4	−164.9	−136.2	**− 78.2**	**− 85.5**	−151.2	−144.6	−114.5	−113.0	−109.6	−126.6
7	−171.9	−152.1	−148.7	−143.3	−120.8	−183.9	−160.2	−149.1	−144.8	−147.6	−152.3
6	−192.0	−163.1	−166.3	−146.3	−151.1	−152.4	−155.7	−152.4	−162.3	−177.5	−171.9
5	−215.0	−180.4	−188.3	−163.8	−162.8	−145.6	−129.5	−156.9	−167.8	−173.2	−159.2
4	−226.0	−202.1	−196.3	−134.4	−159.2	−161.9	−130.5	−158.6	−168.7	−163.8	−170.1
3	−233.9	−211.4	−216.4	−139.7	−153.7	−169.9	−173.9	−182.4	−170.7	−163.8	−190.6
2	−259.9	−239.0	−231.5	−210.0	−168.6	−140.3	−146.9	−125.5	−103.2	− 77.7	−172.3
1	**−140.4**	**−125.2**	**−125.8**	**−109.7**	**− 91.9**	**− 62.1**	**− 45.5**	**− 34.6**	**− 18.5**	**− 24.4**	− 77.8
0	....	....	....	....	....	....	....	....	....	....	....
I_1	−309.1	−315.6	−318.2	−325.0	−325.8	−343.3	−327.3	−325.4	−331.2	−326.8	−313.1
I_2	−292.0	−297.1	−292.7	−318.7	−328.1	−318.9	−312.5	−334.7	−286.6	−248.0	−283.1
C	−204.3	−184.4	−178.5	−156.3	−152.7	−167.5	−148.8	−148.5	−147.2	−144.8	−157.1

The Case I_1 of low temperatures in the isothermal region shows more rapid radiation than Case I_2 of high temperatures, but both indicate about twice as much radiation as in the convectional region *C*. There are two special sources of heat, (1) in the upper cirrus region, and (2) at the surface of the earth, or in the lower cumulus region. The incoming radiation of the sun is divided into two nearly equal parts, the effective radiation penetrating to the surface, but diminishing in intensity in and below the cirrus region, 30 to 10 kilometers, and the returning radiation neutralizing an equal amount of the solar radiation. Pyrheliometer observations at the surface give 2.00 calories per square centimeter per minute, and the bolometer observations probably require the solar constant of 4.00 calories. Hence 2.00 calories are penetrating to the surface and 2.00 calories are returning to space.

Several Important Conclusions

From the formulas,

$$g\,(z_1 - z_0) = -\frac{P_1 - P_0}{\rho_{10}} - \tfrac{1}{2}\,(q_1^2 - q_0^2) - (Q_1 - Q_0),$$

$$g\,\rho_{10}\,(z_1 - z_0) = -\,(P_1 - P_0) - \tfrac{1}{2}\,\rho_{10}\,(q_1^2 - q_0^2) - \rho_{10}\,(Q_1 - Q_0),$$

we must lay down several propositions.

(1) There is properly no such a thing as purely dynamic meteorology, which is defined as a balance between the three terms,

$$g\,(z_1 - z_0) = -\frac{P_1 - P_0}{\rho_{10}} - \tfrac{1}{2}\,(q_1^2 - q_0^2),$$

since these conditions are fulfilled in nature only under temporary circumstances.

(2) In *Margules'* paper, "Ueber die Energie der Sturme," Jahrbuch der K. K. Central-Anstalt für Meteorologie und Erdmagnetismus, Wien, 1903, the formulas are nearly all adiabatic, the gas coefficient *R* and the specific heat C_p are constant, so that the radiation term $(Q_1 - Q_0)$ cannot be computed, and there is no balance possible among the other terms.

(3) In *V. Bjerknes'* paper, "Dynamic Meteorology and

Hydrography," Carnegie Institution of Washington, D. C., 1910, the density is computed by formula (175) instead of by (176), and it is, therefore, a mixed system, since the pressure depends upon (172), and R, Cp, are taken constant, so that there is no theoretical circulation and radiation to be computed.

(4) In *Gold's* paper, "The Isothermal Layer of the Atmosphere and Atmospheric Radiation," Proceedings of the Royal Society, A, Vol. 82, the assumption is made that the mass is proportional to the pressure. This omits the important terms $\frac{1}{2}\,\rho_{10}\,(q_1^2 - q_0^2)$, $\rho_{10}\,(Q_1 - Q_0)$, involving the circulation and radiation, so that the dependent formulas are not properly supported, because the adiabatic case is in reality assumed.

(5) There is a very large literature in meteorology based upon attempts to make a balance of the equation,

$$g\,(z_1 - z_0) = -\frac{P_1 - P_0}{\rho_{10}} - \tfrac{1}{2}\,(q_1^2 - q_0^2),$$

but it is in reality without substantial importance. In spite of the serious difficulty that exists in determining the $(Q_1 - Q_0)$ term, it is necessary that this should be done. There are several large observatories for balloon and kite ascensions which record pressure and temperature, but not humidity and wind velocity, and these entirely fail of their purpose in advancing the interests of meteorology.

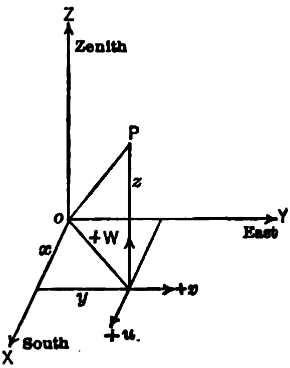
(6) In spite of the fact that only thermodynamic meteorology can have any permanent value in science, there are yet many subordinate problems in the atmosphere which are to be studied without the heat term, such as the stream lines of the circulation, however their forces may have been developed. Accordingly, we proceed, under dynamic meteorology, to deduce the general equations of motion, in the rectangular, the cylindrical, and the polar co-ordinate systems, together with several minor terms, in order to study their application in local storms and circulations of various types.

CHAPTER III

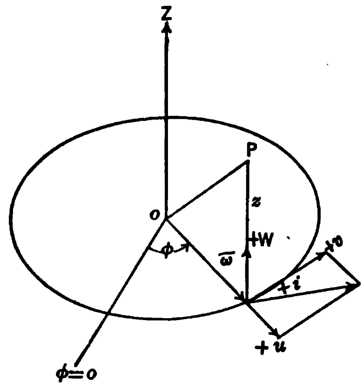
The Hydrodynamics of the Atmosphere

The Co-ordinate Axes

The general equations of motion will be assumed from hydrodynamics, because they are well known, and the proof is

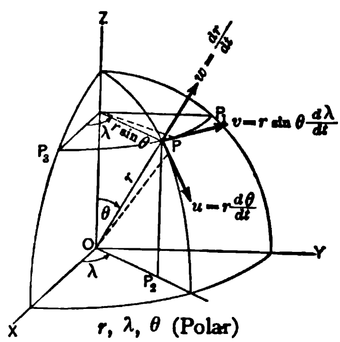


x, y, z (Rectangular)



ϖ, ϕ, z (Cylindrical)

accessible in many treatises, but the equations needed in meteorology will be deduced from them as briefly as possible. There are three systems of co-ordinates, the rectangular, the cylindrical, and the polar, as represented in the diagrams of Fig. 14.



r, λ, θ (Polar)

FIG. 14. Three systems of co-ordinate axes. Compare Fig. 2.

Starting at the point O as the origin of co-ordinates, one can reach the point P directly along the line O P, or indirectly, in rectangular co-ordinates along the distances x, y, z, in succession parallel to the axes; in cylindrical co-ordinates along the radius ϖ, through an angle ϕ counted from an initial line of reference where $\phi = 0$, and along z to P; in polar co-ordinates along a line r whose position in space is determined by the angle λ in the plane $x\,y$ counted from an initial line at the axis of x on that plane, and in the plane $z\,\nu$ at the angular distance θ from the axis of rotation z. These systems are convenient in different problems and must each be developed.

The Co-ordinate Velocities

If q is the velocity along the line $O\,P$ with which a mass is moving, the *co-ordinate velocities* are as follows:

Rectangular	*Cylindrical*	*Polar*
(352) $u = \frac{dx}{dt}$.	(353) $u = \frac{d\varpi}{dt}$.	(354) $u = r\frac{d\theta}{dt}$.
$v = \frac{dy}{dt}$.	$v = \varpi\frac{d\phi}{dt}$.	$v = r\sin\theta\frac{d\lambda}{dt}$.
$w = \frac{dz}{dt}$.	$w = \frac{dz}{dt}$.	$w = \frac{dr}{dt}$.

The Co-ordinate Accelerations

(355) $\dot{u} = \frac{du}{dt} = \frac{d^2x}{dt^2}$.	(356) $u = \frac{d^2\varpi}{dt^2}$.	(357) $\dot{u} = \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)$.
$\dot{v} = \frac{dv}{dt} = \frac{d^2y}{dt^2}$.	$v = \frac{d}{dt}\left(\varpi\frac{d\phi}{dt}\right)$	$\dot{v} = \frac{d}{dt}\left(r\sin\theta\frac{d\lambda}{dt}\right)$.
$\dot{w} = \frac{dw}{dt} = \frac{d^2z}{dt^2}$.	$\dot{w} = \frac{d^2z}{dt^2}$.	$\dot{w} = \frac{d^2r}{dt^2}$.

The Constituents of the Force in Any Direction

Force is measured by the acceleration of a mass in any direction, because it takes force to change the velocity of the mass m at a given point. This will be taken as unity, $m = 1$, in the preliminary equations. A mass of gas or liquid can undergo changes in inertia, changes in volume by expansion or contraction, and changes in figure by rotation, and these three types of forces must be placed in the general equations of motion. The causes that produce these forces in the atmosphere of the earth are external and internal. The external forces are due to gravitation or the potential changes with position; the internal forces are due to pressures which vary in different directions, and cause motion during the restoration to normal equilibrium. The primary source of these pressure forces is the distribution of the thermal energy derived from the solar radiation, or transported in currents of circulation.

The Force of Inertia

This is a partial differential of the velocity in the direction of the co-ordinate-axes, and is

(358) $$\frac{\partial u}{\partial t}, \quad \frac{\partial v}{\partial t}, \quad \frac{\partial w}{\partial t},$$

in each of the co-ordinate systems.

The Forces of Expansion or Contraction

It is convenient to express the differentials of the linear displacements in each co-ordinate system, referred to the x, y, z system.

Rectangular		*Cylindrical*		*Polar*
(359)	$\partial x = \partial x.$	(360)	$\partial x = \partial \varpi.$	(361) $\partial x = r \partial \theta$
	$\partial y = \partial y.$		$\partial y = \varpi \partial \phi.$	$\partial y = r \sin \theta \partial \lambda.$
	$\partial z = \partial z.$		$\partial z = \partial z.$	$\partial z = \partial r.$

Using these values of ∂x, ∂y, ∂z, we obtain for the accelerations due to expansion or contraction:

Rectangular *Cylindrical*

$$(362)\quad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}. \qquad (363)\quad u\frac{\partial u}{\partial \varpi}+v\frac{\partial v}{\varpi\partial \phi}+w\frac{\partial u}{\partial z}.$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}. \qquad u\frac{\partial v}{\partial \varpi}+v\frac{\partial v}{\varpi\partial \phi}+w\frac{\partial v}{\partial z}.$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}. \qquad u\frac{\partial w}{\partial \varpi}+v\frac{\partial w}{\varpi\partial \phi}+w\frac{\partial w}{\partial z}.$$

Polar

$$(364)\quad u\frac{\partial u}{r\partial \theta}+v\frac{\partial u}{r\sin\theta\,\partial\lambda}+w\frac{\partial u}{\partial r}.$$

$$u\frac{\partial v}{r\partial \theta}+v\frac{\partial v}{r\sin\theta\,\partial\lambda}+w\frac{\partial v}{\partial r}.$$

$$u\frac{\partial w}{r\partial \theta}+v\frac{\partial w}{r\sin\theta\,\partial\lambda}+w\frac{\partial w}{\partial r}.$$

In order to give some practical idea of the meaning of these terms, the following example is taken from the Cottage City water-spout, without explanation.

TABLE 44

THE RADII ϖ AND VELOCITIES u, v, w, IN (1) OUTER AND (2) INNER TUBES

Height	Radius ϖ (1)	Radius ϖ (2)	Velocity u (1)	Velocity u (2)	Velocity v (1)	Velocity v (2)	Velocity w (1)	Velocity w (2)
az 90°	83.3	51.9	0	0	11.52	18.49	0.77	1.98
80°	84.0	52.3	− 2.00	− 3.21	11.34	18.20	0.76	1.95
70°	86.0	53.6	− 3.94	− 6.32	10.82	17.37	0.72	1.86
60°	89.6	55.8	− 5.76	− 9.24	9.97	16.01	0.66	1.71
50°	95.2	59.3	− 7.40	−11.88	8.82	14.16	0.58	1.52
40°	103.9	64.8	− 8.82	−14.16	7.40	11.88	0.50	1.27
30°	117.9	73.4	− 9.97	−16.01	5.76	9.24	0.38	0.99
20°	142.5	88.8	−10.82	−17.37	3.94	6.32	0.26	0.68
10°	200.0	124.6	−11.34	−18.20	2.00	3.21	0.13	0.34
0°	∞	∞	−11.52	−18.49	0	0	0	0

It is seen that the velocities in meters per second in the lower half of the dumb-bell vortex of this water-spout undergo changes from the vortex tube (1) to the vortex tube (2), and that as the radius ϖ and the height az change, these velocities change by certain laws. By taking the differences ∂u, ∂v, ∂w, $\partial \varpi$,

$\varpi\,\partial\phi$, ∂z in cylindrical co-ordinates, and using the mean values of u_m, v_m, w_m, the forces that caused these changes from point to point in the vortex can be computed.

The Forces of Rotation

If a mass moves relatively to fixed axes without rotation there are no forces other than inertia and compression or expansion. If the mass also rotates, a new set of forces of rotation is introduced which may be analyzed as follows: Assume that there is a set of rectangular axes fixed in space, and that another set of

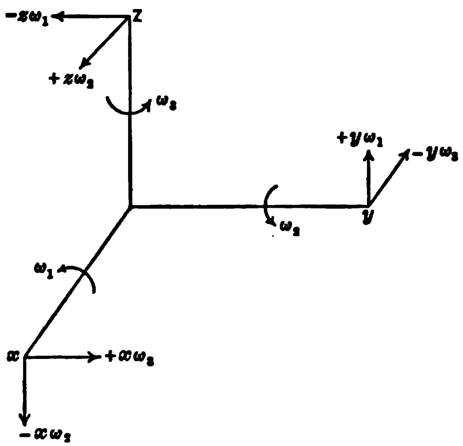


FIG. 15. Angular velocities of motion about fixed axes.

axes attached to the rotating body moves relatively to the fixed axes with the co-ordinate angular velocities ω_1 about the axis x, ω_2 about the axis y, and ω_3 about the axis z. The entire system of rotational velocities at the distances x, y, z, from the origin of rotation is shown on Fig. 15.

The right-handed rotation is defined as that in which a right-handed screw is turned to advance while the axis x moves toward y, the axis y toward z, the axis z toward x, in translation along z, x, y, respectively, and all co-ordinates, of fixed and moving axes, are so related. Thus, in rotation about z, with the angular velocity, ω_3, at the distance x there is instantaneous velocity $+ x\,\omega_3$ parallel to the axis y, and at the distance y there is instantaneous velocity $- y\,\omega_3$ parallel to the axis x, that is, in the negative direction. The same considerations give the component linear velocities parallel to the axes.

(365) Parallel to the axis x, $- y\,\omega_3 + z\,\omega_2$,
Parallel to the axis y, $- z\,\omega_1 + x\,\omega_3$,
Parallel to the axis z, $- x\,\omega_2 + y + y\,\omega_1$.

These symbols are all arranged in the cyclic order, and are easily verified from Fig. 15.

Similarly the component accelerations are found by substituting u, v, w, for x, y, z, in succession, and we have the accelerations parallel to the axes:

(366) Parallel to the axis x, $- v\,\omega_3 + w\,\omega_2$,
Parallel to the axis y, $- w\,\omega_1 + u\,\omega_3$,
Parallel to the axis z, $- u\,\omega_2 + v\,\omega_1$.

Since these forms are entirely general, it is only necessary to substitute the special values of $\omega_1, \omega_2, \omega_3$, for given cases, to apply the formulas to particular problems. If only fixed axes are employed, we have $\omega_1 = 0$, $\omega_2 = 0$, $\omega_3 = 0$. If cylindrical axes are employed, there is rotation about the axis z only, so that $\omega_1 = 0$, $\omega_2 = 0$, $\omega_3 = + \frac{u}{\varpi}$. If polar co-ordinates are employed the angular velocities become $\omega_1 = - \frac{v}{r}$, $\omega_2 = + \frac{u}{r}$, $\omega_3 = + \frac{v}{r \tan \theta}$. Placing these results in tabular form, we have, as can be seen by the definition of angular velocity,

Rectangular Fixed Axes	*Cylindrical Co-ordinates*	*Polar Co-ordinates*
(367) $\omega_1 = 0.$	(368) $\omega_1 = 0.$	(369) $\omega_1 = - \frac{v}{r}.$
$\omega_2 = 0.$	$\omega_2 = 0.$	$\omega_2 = + \frac{u}{r}.$
$\omega_3 = 0.$	$\omega_3 = + \frac{v}{\varpi}.$	$\omega'_3 = + \frac{v}{r \tan \theta}.$

In cylindrical co-ordinates the angle increases with the angular velocity $\omega_3 = \frac{v}{\varpi}$. In polar co-ordinates the angular velocities may be illustrated by Fig. 16.

Let a mass move from P_0 to P_3 relative to the rotating earth, whose axis of rotation is $O\,Z$. Lay down the fixed axes x, y, z,

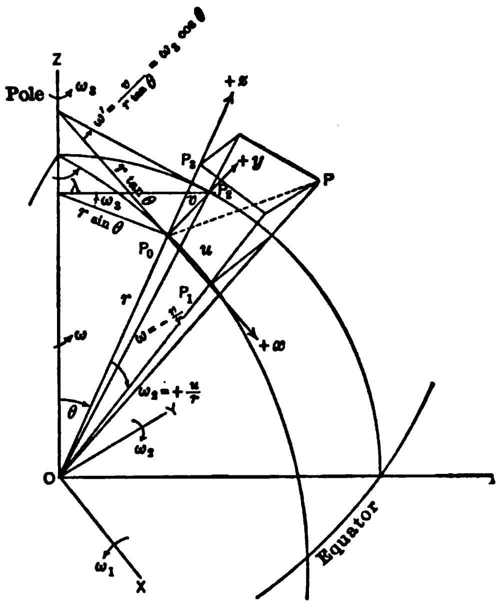


FIG. 16. Angular velocities of moving axes relative to fixed axes.

at the center O, draw the radius r to P_0 at the polar distance θ, in longitude λ counted from an initial meridian. The angular velocity of the rotating earth is ω_3; $r \sin \theta$ is the perpendicular distance of P_0 from $O\,Z$, and $r \tan \theta$ is the tangential distance from P_0 to $O\,Z$. If the mass moves from P_0 to P_1 it rotates about the axis y with the angular velocity $\omega_2 = \frac{u}{r}$, since it moves in the positive direction of x; if it moves from P_0 to P_3 it rotates about z with the angular velocity $\omega'_3 = \frac{v}{r \tan \theta}$; if it moves from

P_0 to P_2 it rotates about x with angular velocity $-\omega_1 = \frac{v}{r}$, because in the motion from P_0 to P_2 the velocity is in the negative direction of rotation. If the mass moves from P_0 to P_3 along the radius there is no rotation. These rotational relations can be rigorously proved by analytical demonstration, but as the analysis is rather complicated the reader is referred to the standard treatises by Basset, Lamb, W. Wien, and others. The formulas for the angular velocities in terms of the linear velocities are as follows:

(370) Rectangular co-ordinates.

$$2\,\omega_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}.$$

$$2\,\omega_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}.$$

$$2\,\omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

(371) Cylindrical co-ordinates.

$$2\,\omega_1 = \frac{\partial \omega}{\varpi\,\partial \phi} - \frac{\partial v}{\partial z}.$$

$$2\,\omega_2 = \frac{\partial u}{\partial z} - \frac{\partial \omega}{\partial \varpi}.$$

$$2\,\omega_3 = \frac{\partial v}{\partial \varpi} - \frac{\partial u}{\varpi\,\partial \phi} + \frac{v}{\varpi}.$$

(372) Polar co-ordinates.

$$2\,\omega_1 = \frac{\partial \omega}{r \sin\theta\,\partial \lambda} - \frac{\partial v}{\partial r} - \frac{v}{r}.$$

$$2\,\omega_2 = \frac{\partial u}{\partial r} - \frac{\partial w}{r\,\partial \theta} + \frac{u}{r}.$$

$$2\,\omega_3 = \frac{\partial v}{r\,\partial \theta} - \frac{\partial u}{r \sin\theta\,\partial \lambda} + \frac{v}{r \tan\theta}.$$

If the linear velocities u, v, ω, at the point x, y, z, are known, the angular velocities ω_1, ω_2, ω_3, can be computed in the three systems of co-ordinates.

It should be observed that since,

$$\omega_3 = \frac{v}{r \sin \theta}, \text{ and } \omega'_3 = \frac{v}{r \tan \theta} = \frac{v \cos \theta}{r \sin \theta},$$

(373) $$\omega'_3 = \omega_3 \cos \theta,$$

so that at the pole for $\theta = 0$, $\cos \theta = 1$, $\omega'_3 = \omega_3$, and at the equator for $\theta = 90°$, $\cos \theta = 0$, $\omega'_3 = 0$. At the pole a point rotates about the axis of rotation with the angular velocity of the rotating earth, but at the equator it does not have angular velocity about z.

The Pressure Gradients

The earth in its rotation, acted upon by the gravitation of its own mass, has assumed the form of an oblate spheroid, being flattened at the poles, and if the atmosphere were not acted upon by thermal forces it would be arranged in layers of density parallel to those of the earth's solid mass. The heating of the atmosphere in the tropics by solar radiation, and secondarily in the other latitudes, disturbs these level surfaces of pressure, by lifting some areas and depressing other areas during the processes of heating, cooling, and circulation. The change in pressure from the normal pressure in a given linear distance is the pressure gradient $-\frac{dP}{ds}$, and since the forces are directed from the higher to the lower pressure the minus sign is used, $-\frac{dP}{ds}$. Finally, the forces are all to be reduced to the unit density so that the forms become, for the three co-ordinate systems, respectively,

	Rectangular Co-ordinates		*Cylindrical Co-ordinates*		*Polar Co-ordinates*
(374)	$-\frac{1}{\rho}\frac{\partial P}{\partial x}$.	(375)	$-\frac{1}{\rho}\frac{\partial P}{\partial \varpi}$.	(376)	$-\frac{1}{\rho}\frac{\partial P}{r\,\partial \theta}$.
	$-\frac{1}{\rho}\frac{\partial P}{\partial y}$.		$-\frac{1}{\rho}\frac{\partial P}{\varpi\,\partial \phi}$.		$-\frac{1}{\rho}\frac{\partial P}{r \sin \theta\,\partial \lambda}$.
	$-\frac{1}{\rho}\frac{\partial P}{\partial z}$.		$-\frac{1}{\rho}\frac{\partial P}{\partial z}$.		$-\frac{1}{\rho}\frac{\partial P}{\partial r}$.

These expressions can be evaluated into many forms, which are convenient for practical computations, as will be shown in a later section.

The Potential Gradient

If the potential of the external forces of attraction of the earth's mass is V, then the forces due to such a potential are,

$$(377) \quad -\frac{\partial V}{\partial x}, \quad -\frac{\partial V}{\partial y}, \quad -\frac{\partial V}{\partial z}.$$

In the case of the earth's mass the forces $-\frac{\partial V}{\partial x}$ and $-\frac{\partial V}{\partial y}$, on the meridians and on the parallels of latitude, respectively, are chiefly concerned with the determination of its existing figure. In the problems of meteorology these forces can be neglected, so that there remains only the vertical potential gradient,

$$(378) \quad -\frac{\partial V}{\partial z} = -g.$$

There has been considerable confusion in the literature of this subject because the positive direction is taken upward by some authors, but downward by others. If the positive direction is upward, as where the positive motion along the radius is outward from the center of the earth, we have

$$(379) \qquad \frac{dV}{dz} = +g, \text{ and } V = +gz.$$

If the positive direction is inward, as along the path of a falling body, we have,

$$(380) \qquad \frac{dV}{dz} = -g, \text{ and } V = -gz.$$

The positive direction upward is used by Ferrel, Sprung, Oberbeck, Basset in some sections, Helmholtz, Bigelow; the positive direction inward is used by Basset in some sections, Lamb, V. Bjerknes.

The force of gravity is. however, modified on the rotating earth by the centrifugal force, which acts only in planes perpendicular to the axis of rotation. These forces can be resolved along and perpendicular to the axis, as follows, since the centrifugal force is $\frac{1}{2}\,\omega_0^2\,\varpi^2$.

The total potential is,

(381) $V = g\,r + \frac{1}{2}\,\omega_0^2\,\varpi^2,$

where ω_0 is the angular velocity, and $\omega_0 = n$ in the notation for the rotating earth. By (66) we have

(382) $g = g_0 \frac{R^2}{r^2}$, and

(383) $g\,r = g_0 \frac{R^2}{r}$, so that

(384) $V = \frac{g_0 R^2}{r} + \frac{1}{2}\,\omega_0^2\,\varpi^2.$

Taking the differential along the axis z, and perpendicular to it along ϖ, there results:

(385) $-\frac{d\,V}{d\,z} = +\,g_0 \frac{R^2}{r^2} \frac{d\,r}{d\,z}.$

(386) $-\frac{d\,V}{d\,\varpi} = +\,g_0 \frac{R^2}{r^2} \frac{d\,r}{d\,\varpi} - \omega_0^2\,\varpi.$

We have:

(387) $r^2 = x^2 + y^2 + z^2 = \varpi^2 + z^2.$ Differentiating,

(388) $2\,r\,d\,r = 2\,\varpi\,d\,\varpi + 2\,z\,d\,z$, and

(389) $\frac{d\,r}{d\,z} = \frac{z}{r}, \quad \frac{d\,r}{d\,\varpi} = \frac{\varpi}{r}.$ Hence, for $R = r$

(390) $-\frac{d\,V}{d\,z} = g \cdot \frac{z}{r} = g \cos\theta.$

(391) $-\frac{d\,V}{d\,\varpi} = g \cdot \frac{\varpi}{r} - \omega_2^2\,\varpi = g \sin\theta - \omega_2^2\,\varpi.$

Equations of Continuity

When a mass of air streams through a given space, as a cubic meter at a given place, as much air must pass out of it as enters it, or else there will be congestion, and a change in the continuity. The equations which finally control any solution of current functions must satisfy these equations of continuity.

(392) *Rectangular Co-ordinates.* $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \text{ for } \rho = \text{const.}$$

(393) $$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0.$$

(394) *Cylindrical Co-ordinates.* $$\frac{\partial(\varpi u)}{\partial \varpi} + \frac{\partial v}{\partial \varphi} + \varpi \frac{\partial w}{\partial z} = \frac{\partial u}{\partial \varpi} + \frac{u}{\varpi} + \frac{\partial w}{\partial z} = 0.$$

(395) *Polar Co-ordinates.* $$r \frac{\partial (u \sin \theta)}{\partial \theta} + r \frac{\partial v}{\partial \lambda} + \sin \theta \frac{\partial (r^2 w)}{\partial r} = 0.$$

The Operator ∇^2

The sum of the second differentials in three co-ordinates is often used in analytical discussions, and the symbol ∇^2 has been adopted for this process.

(396) *Rectangular Co-ordinates.* $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

(397) *Cylindrical Co-ordinates.* $$\nabla^2 = \frac{\partial^2}{\partial \varpi^2} + \frac{1}{\varpi}\frac{\partial}{\partial \varpi} + \frac{1}{\varpi^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}.$$

(398) *Polar Co-ordinates.* $$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2}{\partial \lambda^2}.$$

The Total Differential $\frac{d}{dt}$.

The symbol $\frac{d}{dt}$ is often used to include the terms of the inertia and expansion or contraction.

(399) $$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}.$$

There are a series of complicated terms used to express the internal forces caused by the stretches, the shears, the dilatation, the tractions due to elasticity and viscosity within the masses, but these will be omitted in this place. They are summarized on pages 499–501 of the Cloud Report.

Summary of the Equations of Motion

By putting together the terms that have now been explained, there result the general equations of motion which are at the basis of all the dynamic meteorology, omitting the heat term $d\,Q$.

Rectangular Co-ordinates

$$(400) \quad -\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - v\,\omega_3 + w\,\omega_2.$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial y} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} - w\,\omega_1 + u\,\omega_3.$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - u\,\omega_2 + v\,\omega_1 + g.$$

Cylindrical Co-ordinates

First form of cylindrical equations

$$(401) \quad -\frac{1}{\rho}\frac{\partial P}{\partial \varpi} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \varpi} + v\frac{\partial u}{\varpi\,\partial \varphi} + w\frac{\partial u}{\partial z} - \frac{v^2}{\varpi}.$$

$$-\frac{1}{\rho}\frac{\partial P}{\varpi\,\partial \varphi} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial \varpi} + v\frac{\partial v}{\varpi\,\partial \varphi} + w\frac{\partial v}{\partial z} + \frac{u\,v}{\varpi}.$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial \varpi} + v\frac{\partial w}{\varpi\,\partial \varphi} + w\frac{\partial w}{\partial z} + g.$$

Polar Co-ordinates

First form of polar equations

$$(402) \quad -\frac{1}{\rho}\frac{\partial P}{\rho\,\partial \theta} = \frac{\partial u}{\partial t} + u\frac{\partial u}{r\,\partial \theta} + v\frac{\partial u}{r\sin\theta\,\partial \lambda} + w\frac{\partial u}{\partial r} - \frac{v^2}{r}\cot\theta + \frac{w\,u}{r}.$$

$$-\frac{1}{\rho}\frac{\partial P}{r\sin\theta.\partial \lambda} = \frac{\partial v}{\partial t} + u\frac{\partial v}{r\,\partial \theta} + v\frac{\partial v}{r\sin\theta\,\partial \lambda} + w\frac{\partial v}{\partial r} + \frac{u\,v}{r}\cot\theta + \frac{w\,v}{r}.$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{\partial w}{\partial t} + u\frac{\partial w}{r\,\partial \theta} + v\frac{\partial w}{r\sin\theta\,\partial \lambda} + w\frac{\partial w}{\partial r} - \frac{u^2}{r} - \frac{v^2}{r} + g.$$

Equations of Motion for the Rotating Earth

The equations of motion for moving axes are further modified when the axes are attached to the earth which is rotating with the constant angular velocity ω_3.

Cylindrical Co-ordinates on the Rotating Earth

The linear velocities remain the same except that the linear velocity eastward is increased by the term $\omega_3 . \varpi \cos \theta$. The angular velocity about z is changed by the addition of the term $\omega_3 \cos \theta$.

(403) Linear velocities, $u' = u.$

$$v' = v + \omega_3 . \varpi \cos \theta.$$

$$w' = w.$$

Angular velocities, $\omega'_1 = 0.$

$$\omega'_2 = 0.$$

$$\omega'_3 = \omega_3 \cos \theta + \frac{v}{\varpi}.$$

The partial differentials are also modified:

(404) $$\frac{\partial u'}{\partial t} = \frac{\partial u}{\partial t}.$$

$$\frac{\partial v'}{\partial t} = \frac{\partial (v + \omega_3 . \varpi \cos \theta)}{\partial t} = \frac{\partial v}{\partial t} + u . \omega_3 \cos \theta.$$

$$\frac{\partial w'}{\partial t} = \frac{\partial w}{\partial t}.$$

Substituting these terms in the general equations (400), and using (399), we have:

(405) $$-\frac{1}{\rho}\frac{\partial P}{\partial \varpi} = \frac{du}{dt} - (\omega_3 . \varpi \cos \theta + v)\left(\omega_3 \cos \theta + \frac{v}{\varpi}\right).$$

$$-\frac{1}{\rho}\frac{\partial P}{\varpi \partial \phi} = \frac{dv}{dt} + u . \omega_3 \cos \theta + u \left(\omega_3 \cos \theta + \frac{v}{\varpi}\right).$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{dw}{dt} + g.$$

Performing the algebraic work, and substituting for the relative angular velocity eastward $\nu = \frac{v}{\varpi}$, we find:

Second form of cylindrical equations

(406)
$$-\frac{1}{\rho}\frac{\partial P}{\partial \varpi} = \frac{du}{dt} - 2\,\omega_3 \cos\theta . v - \frac{v^2}{\varpi} = \frac{du}{dt} - (2\omega_3 \cos\theta + \nu)\, v.$$

$$-\frac{1}{\rho}\frac{\partial P}{\varpi \partial \phi} = \frac{dv}{dt} + 2\omega_3 \cos\theta . u + \frac{uv}{\varpi} = \frac{dv}{dt} + (2\,\omega_3 \cos\theta + \nu)\, u.$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{dw}{dt} + g \qquad = \frac{dw}{dt} + g.$$

Polar Co-ordinates on the Rotating Earth

The linear velocity eastward is increased by the term $\omega_3 r \sin\theta$, and consequently the angular velocities ω_1 and ω_3 are modified to conform with it.

(407) Linear velocities, southward, $u' = u$.

eastward, $v' = v + \omega_3 . r \sin\theta$.

zenithward, $w' = w$.

(408) Angular velocities, about the axis x, $\omega'_1 = -\frac{v + \omega_3 r \sin\theta}{r}$.

axis y, $\omega'_2 = +\frac{u}{r}$.

axis z, $\omega'_3 = +\frac{v + \omega_3 r \sin\theta}{r \tan\theta}$.

(409) The partial differentials $\frac{\partial u'}{\partial t} = \frac{\partial u}{\partial t}$.

$$\frac{\partial v'}{\partial t} = \frac{\partial (v + \omega_3 r \sin\theta)}{\partial t} = \frac{\partial v}{\partial t} + u . \omega_3 \cos\theta + w . \omega_3 \sin\theta.$$

$$\frac{\partial w'}{\partial t} = \frac{\partial w}{\partial t}.$$

Substituting these values in the equations (400), we have,

$$(410)\quad -\frac{1}{\rho}\frac{\partial P}{r\partial\theta} = \frac{du}{dt} - (v + \omega_3 r\sin\theta)\left(\frac{v + \omega_3 r\sin\theta}{r\tan\theta}\right) + w\frac{u}{r}.$$

$$-\frac{1}{\rho}\frac{\partial P}{r\sin\theta\,\partial\lambda} = \frac{dv}{dt} + w\left(\frac{v + \omega_3 r\sin\theta}{r}\right) + u\left(\frac{v + \omega_3 r\sin\theta}{r\tan\theta}\right).$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{dw}{dt} - u\,.\,\frac{u}{r} - (v + \omega_3 r\sin\theta)\left(\frac{v + \omega_3 r\sin\theta}{r}\right) + g.$$

Performing the multiplications and reductions,

Second form of polar equations

$$(411)\quad -\frac{1}{\rho}\frac{\partial P}{r\partial\theta} = \frac{du}{dt} - \frac{v^2}{r}\cot\theta + \frac{uw}{r} - 2\,\omega_3\cos\theta\, v \;.\;.\;.\; - r\,\omega_3^2\sin\theta\cos\theta.$$

$$-\frac{1}{\rho}\frac{\partial P}{r\sin\theta\,\partial\lambda} = \frac{dv}{dt} + \frac{uv}{r}\cot\theta + \frac{wv}{r} + 2\,\omega_3\cos\theta\,.\,u + 2\,\omega_3\sin\theta\,.\,w.\;.\;.\;.$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{dw}{dt}\;.\;.\;.\; - \frac{u^2 + v^2}{r} - 2\,\omega_3\sin\theta\,.\,v\;.\;.\;.\; - r\,\omega_3^2\sin^2\theta.$$

The terms in $2\,\omega_3$ represent the deflecting forces due to the earth's rotation, which always act at right angles to the linear velocity q with which a mass is moving in any direction. They deflect a moving body to the right in the northern hemisphere, but to the left in the southern hemisphere. The deflecting force is a maximum and equal to $2\,\omega_3\,q$ on the horizontal plane at the poles; it is equal to zero at the equator, for all velocities in the horizontal plane. If the velocity is vertical the deflecting force due to this term is zero at the pole and equal to $2\,\omega_3\,w$ at the equator. The terms in $r\,\omega_3^2$ represent the forces which change the figure of the earth from a sphere to an oblate spheroid.

They need not be considered in practical meteorology, but are important in geodesy.

The derivation of the general equations of motion given above is exceedingly simple and direct, showing immediately where all the terms come from which are concerned with general motions. They are equally true for the sun, the earth, and the planets, and all observed motions must conform to them.

Connection Between the General Equations of Motion and the Thermal Equations of Energy

We found that with changes of the temperature gradient the variation of the pressure $\frac{dP}{\rho}$ is expressed by equation (190), when there is no change in heat Q not otherwise accounted for in this process of motion. But in case all the heat energy is not expended in motion, as where a part escapes in radiation or in internal molecular or atomic motions, as in ionization, a new term must be added to take this into the account, so that

$$(412)\quad \frac{\partial P}{\rho} = -J + n\,C_p\,dT + C_p\,T\log T \;.\; dn.$$

In order to avoid confusion of symbols in this set of equations, take $\nu = \omega_2$ the angular velocity of the rotating earth, and combine the equations (411) and (412). It should be noted that since $\nu = \frac{v}{r\sin\theta}$, by Fig. 16, we have

$$(413)\quad \frac{v}{r} = \nu\sin\theta,\quad \frac{v^2\cot\theta}{r} = v\cos\theta \;.\; \nu,\quad \frac{uv\cot\theta}{r} = u\cos\theta \;.\; \nu,$$

and we shall make use of them in the following transformations. In order to connect together the hydrodynamic and the thermodynamic systems by the law of the conservation of energy, we have $\frac{dP}{\rho}$ the same in both systems. Combining these terms and making the reductions, substituting dx, dy, dz for the corresponding expressions of the linear displacements in polar co-ordinates, and adding dQ for the change in the heat contents, and $d\,J$ for the energy in the form of electric and magnetic forces, there result for the forces of acceleration,

The General Hydrodynamic and Thermodynamic Equations of Motion

Third form of polar equations

(414)
$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{d\,u}{d\,t} - \cos\theta\,(2\,\omega_3 + \nu)\,v + \frac{u\,w}{r} + \frac{\partial(Q+J)}{\partial x} = -\,Cp\,\frac{\partial T}{\partial x} - Cp\,T\log T\,\frac{\partial n}{\partial x}.$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial y} = \frac{d\,v}{d\,t} + \cos\,\theta\,(2\,\omega_3 + \nu)\,u + \sin\,\theta\,(2\,\omega_3 + \nu)\,w + \frac{\partial(Q+J)}{\partial y} = -\,Cp\,\frac{\partial T}{\partial y} - Cp\,T\log T\,\frac{\partial n}{\partial y}.$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{d\,w}{d\,t} - \sin\theta\,(2\,\omega_3 + \nu)\,v - \frac{u^2}{r} + g + \frac{\partial(Q+J)}{\partial z} = -\,Cp\,\frac{\partial T}{\partial z} - Cp\,T\log T\,\frac{\partial n}{\partial z}.$$

Corresponding with these co-ordinate equations is the differential equation,

(415)
$$-\frac{d\,P}{\rho\,d\,r} = g + q\frac{d\,q}{d\,r} + \frac{d\,Q}{d\,r} + \frac{d\,J}{d\,r} = -\,Cp\,\frac{d\,T}{d\,r} - Cp\,T\log T\,\frac{d\,n}{d\,r}.$$

This equation has been already discussed in its vertical variations, but the more difficult task is to explain its meaning in the horizontal directions $(x,\ y)$. The terms $\frac{d\,u}{d\,t}$, $\frac{d\,v}{d\,t}$, $\frac{d\,w}{d\,t}$ contain the inertia and the forces of expansion and contraction as expressed in equations (358)–(364), (396)–(398), and these must generally be employed in the study of tornadoes, water-spouts, hurricanes, ocean and land cyclones. The terms in J can hardly be discussed until the subjects of absorption in the spectrum, scattering in the atmosphere, electric and magnetic forces can be more thoroughly worked out. The difficulties in determining the value of Q have already been stated.

The Equations for the Work of Circulation

If the equations for the force of *acceleration* (414) are multiplied by $d\,x$, $d\,y$, $d\,z$, respectively, that is, if the force is multiplied by the distance through which it acts, they give the *work* expended in transporting the mass from one point to another. Still retaining the unit mass, $m = 1$, the work-equations become,

$$(415)\quad -\frac{\partial P}{\rho} = d\,u\frac{d\,x}{d\,t} - \cos\theta\,(2\,\omega_3 + \nu)\,v\,d\,x + \frac{u\,w}{r}d\,x \ldots + \frac{\partial(Q+J)}{\partial\,x}\,d\,x.$$

$$-\frac{\partial P}{\rho} = d\,v\frac{d\,y}{d\,t} + \cos\theta\,(2\,\omega_3 + \nu)\,u\,d\,y + \sin\theta\,(2\,\omega_3 + \nu)\,w\,d\,y + \frac{\partial(Q+J)}{\partial\,y}\,d\,y.$$

$$-\frac{\partial P}{\rho} = d\,w\frac{d\,z}{d\,t} - \sin\theta\,(2\,\omega_3 + \nu)\,v\,d\,z - \frac{u\,u\,d\,z}{r} + g\,d\,z + \frac{\partial(Q+J)}{\partial\,z}\,d\,z.$$

We have the auxiliary equations,

$$(416)\quad v\,d\,x = u\,d\,y,\quad w\,d\,x = u\,d\,z,\quad w\,d\,y = v\,d\,z,$$

and it is seen that by substitution all the terms cancel except the following, after using the total differential,

$$(417)\quad -\frac{d\,P}{\rho} = u\,d\,u + v\,d\,v + w\,d\,w + g\,d\,z + d\,(Q+J).$$

The integral equation is known as Bernoulli's,

$$(418)\quad -\int\frac{d\,P}{\rho} = \tfrac{1}{2}\,(u^2+v^2+w^2) + \int g\,d\,z + (Q+J),\ \text{one limit,}$$

$$(419)\quad -\frac{P_1 - P_0}{\rho_{10}} = \tfrac{1}{2}\,(q_1^2 - q_0^2) + g\,(z_1 - z_0) + (Q_1 - Q_0) + (J_1 - J_0),\ \text{two limits.}$$

The other combination, in (414), gives,

$$(420) \quad -\int \frac{dP}{\rho} = -\int n\,Cp\,dT + \int Cp\,T \log T\,dn. \text{ as (202).}$$

In case $J - J_0 = 0$, we have, in practice, for the static pressure P' and the dynamic pressure P'', if $P = P' + P''$.

$$(421) \quad -\int \frac{dP'}{\rho} = g\,(z_1 - z_0) = -Cp\,n_1\,(T_1 - T_0), \text{ static pressure,}$$

$$(422) \quad -\int \frac{dP''}{\rho} = \tfrac{1}{2}\,(q_1^2 - q_0^2) + (Q_1 - Q_0), \text{ circulation and radiation,}$$

$$= (Cp_a - Cp_{10})\,(T_a - T_0) = n_1\,(Cp_a - Cp_{10})\,(T_1 - T_0).$$

The equation (421) has already been discussed under Barometry (39) with others; also, under gradients (12) with others. The equation (422) remains to be considered under circulation and the variation of the gradients.

The Evaluation of the Term $-\int \frac{dP}{\rho}$.

There are several methods of treating the computations involved in the thermodynamic and static equation (419), which considers the circulation and the vertical pressure along with the thermodynamic sources of the pressure differences. Since all the terms in (415), which represent the deflecting and the centrifugal forces, disappeared on the substitution of the values in (416), it follows that these forces are always acting at right angles to the direction of the motion of the mass moving with the velocity q in any direction relatively to the surface of the rotating earth. The deflecting and the centrifugal forces have, therefore, no effect upon the work of the circulation of the atmosphere, any more than the central forces do upon the work of the motion of a planet in its orbit. They change the direction of the motion by the composition of the forces, but it requires no additional pressure to overcome these forces at right angles to the path of motion of a current of air.

I. It is convenient to evaluate the term $-\int \frac{dP}{\rho}$ in terms of the temperature, eliminating the density. This can be done by substituting from (62),

(423) $$\frac{1}{\rho} = \frac{1}{\rho_0} \frac{P_0}{P} \frac{T}{T_0} = \frac{g_0 \rho_0 l_0}{\rho_0 T_0} \cdot \frac{T}{P} = \frac{g_0 l_0}{T_0} \cdot \frac{T}{P}.$$

Substituting and using the values of the constants in Table 3, omitting henceforward $(Q_1 - Q_0)$,

(424) $$-\int \frac{dP}{P} = \frac{T_0}{g_0 l_0} \cdot \frac{1}{2T} (q^2 - q_0^2) + \frac{T_0}{g_0 l_0} \cdot \frac{g}{T} (z - z_0).$$

Integrating between limits for natural logarithms,

(425) $$\log P_0 - \log P = \frac{1}{574.067\, T} (q^2 - q_0^2) + \frac{g}{287.033\, T} (z - z_0).$$

(426) $$\log P_0 - \log P = \frac{1}{156720} \cdot \frac{T_0}{T} (q^2 - q_0^2) + \frac{1}{7991.04} \cdot \frac{T_0}{T} \cdot \frac{g}{g_0} (z - z_0).$$

The same formulas in common logarithms become.

(427) $$\log P_0 - \log P = \frac{1}{1321.837\, T} (q^2 - q_0^2) + \frac{g}{660.919\, T} (z - z_0).$$

(428) $$\log P_0 - \log P = \frac{1}{360862} \cdot \frac{T_0}{T} (q^2 - q_0^2) + \frac{1}{18400} \cdot \frac{T_0}{T} \cdot \frac{g}{g_0} (z - z_0).$$

(429) The constants are derived in succession:

$$574.067 = \frac{2 g_0 l_0}{T_0}. \qquad 287.033 = \frac{g_0 l_0}{T_0}.$$

$$156720 = 2 g_0 l_0. \qquad 7991.04 = l_0.$$

$$1321.837 = \frac{2 g_0 l_0}{T_0 M}. \qquad 660.919 = \frac{g_0 l_0}{T_0 M}.$$

$$360862 = \frac{2 g_0 l_0}{M}. \qquad 18400 = \frac{l_0}{M}.$$

The second term of (428) is the same as in formula (159), to which static barometry is usually confined, but the term in the velocity should be added for accurate computations. In the integrations between two points the mean temperature, $T_{10} = T_m$, and the mean gravity, $g_{10} = g_m$, of the air column should be employed, as already explained.

II. Since by the Boyle-Gay Lussac Law, we have:

$$\frac{1}{\rho} = \frac{RT}{P},$$

this value can be introduced into the equation (419), so that

$$(430) \quad -\int \frac{dP}{P} = \frac{1}{2RT}(q^2 - q_0^2) + \frac{g}{RT}(z - z_0).$$

This is correct because $2R = 574.067$, by Table 3, and $\frac{g_0}{RT_0} = \frac{1}{l} = \frac{1}{7991.04}$, and these satisfy (425), (426).

III. Since, by equations (176) and (178), we have, for $n = 1$,

$$(431) \quad \frac{1}{\rho} = \frac{1}{\rho_0}\left(\frac{P_0}{P}\right)^{1/k} = \frac{R_0 T_0}{P_0}\left(\frac{P_0}{P}\right)^{1/k} = R_0 T_0 P_0^{\frac{1-k}{k}} P^{-\frac{1}{k}},$$

this value can be introduced into the equation (419). Then,

$$(432) \quad -\int \frac{dP}{P} = \frac{1}{R_0 T_0}\left(\frac{P}{P_0}\right)^{\frac{k-1}{k}} [\tfrac{1}{2}(q^2 - q_0^2) + g(z - z_0)].$$

Since $\left(\frac{P}{P_0}\right)^{\frac{k-1}{k}} = \left(\frac{T}{T_0}\right)^n$, we find again, for $n = 1$,

$$(433) \quad -\int \frac{dP}{P} = \frac{1}{R_0 T_0}[\tfrac{1}{2}(q^2 - q_0^2) + g(z - z_0)].$$

In case n is not equal to unity, and the non-adiabatic temperature distribution of the air is considered, we have:

$$(434) \quad -\int \frac{dP}{P} = \frac{T^{n-1}}{R_0 T_0^n}[\tfrac{1}{2}(q^2 - q_0^2) + g(z - z_0)].$$

IV. In case it is desired to reduce the equation of motion to a form where the standard density is unity, we shall have, $\rho_0 = 1$, and

(435) $\frac{1}{\rho} = P_0^{1/k} . P^{-1/k} = (R_0 T_0)^{1/k} . \frac{P^{\frac{k-1}{k}}}{P}$. Hence,

(436) $-\int \frac{dP}{P} = \frac{P^{\frac{1-k}{k}}}{(R_0 T_0)^{1/k}} [\frac{1}{2} (q^2 - q_0^2) + g (z - z_0)].$

This form eliminates the temperature and throws all the discussion upon P, being an impure form, since P can be found only by trials.

V. It is often desirable to integrate the term $-\int \frac{dP}{\rho}$ without using the mean temperature of the column, and this can be done by determining the mean density of the column ρ_m. Then,

(437) $-\int \frac{dP}{\rho} = -\frac{P_1 - P_0}{\rho_m} = \frac{1}{2} (q^2 - q_0^2) + g (z - z_0) + (Q_1 - Q_0)$, and,

(438) $-\frac{P_1 - P_0}{\rho_m} = + n_1 (Cp_a - Cp_{10}) (T_1 - T_0) - n_1 Cp_{10} (T_1 - T_0).$

Numerical Check on the Two Systems of Formulas

We have already found, if the humidity, gravity, and mountain terms are now omitted, that

(439) $\log \frac{P_0}{P} = \log \frac{B_0}{B} = \frac{z - z_0}{18400 + 67.6\,\theta}$, by (159), and that

(440) $\log \frac{P_0}{P} = \log \frac{B_0}{B} = -n \frac{k}{k-1} (\log T - \log T_0)$, by (182),

so that the two methods of computation, through the mean temperature of the air column θ and the gradient ratio n can be compared and checked. They prove by trial examples to be in agreement.

Numerical Evaluations of the Pressure Gradient

It is necessary to reduce the difference on the barometric pressure, at any distance apart on a horizontal level, to a standard distance which is taken as 1 degree = 111 111 meters on the surface of the earth.

G = the barometric difference for the distance 111 111 meters.

D_0 = 1 degree or 111 111 meters on the surface of the earth.

D_1 = any distance on the surface.

$$(441)\quad \frac{G}{D_0} = \frac{G}{111\,111} = \frac{dB}{dx} = \frac{B_0 - B}{D_0} = \frac{B_1 - B}{D_1}. \quad \text{Hence,}$$

$$(442)\quad G = \frac{D_0}{D_1}(B_1 - B).$$

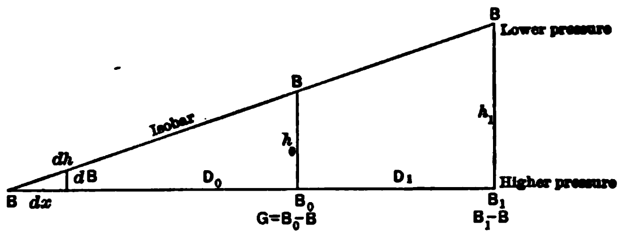


FIG. 17. The derived gradient for the distance of 1 degree.

The term $-\frac{1}{\rho}\frac{\partial P}{\partial x}$ can be evaluated in many ways.

Since $P = g_0\, \rho_m\, B = g_0\, p$, we have:

$$(443)\quad \frac{\partial P}{\partial x} = g_0\ \rho_m \frac{\partial B}{\partial x} = g_0 \frac{\partial p}{\partial x}.$$

$$(444)\quad \frac{\partial P}{\partial x} = g_0\, \rho_m \frac{G}{111\,111} = 1.200\, G \text{ (meter)} = 0.0012\, G \text{ (mm.)}.$$

$$(445)\quad \frac{dp}{dx} = \rho_m \frac{G}{111\,111} = 0.12236\, G \text{ (meter)} = 0.00012236\, G \text{ (mm.)}.$$

Hence, from these formulas and others preceding,

$$(446)\quad -\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{P_0}{\rho_0}\frac{T}{T_0 P}\frac{\partial P}{\partial x} = -g_0\, l \frac{T}{T_0}\frac{\partial \log P}{\partial x} = -\frac{g_0\, \rho_m\, B_n}{\rho_0}\frac{T}{T_0}\frac{\partial \log P}{\partial x}.$$

$$(447)\quad -\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{1}{\rho} g_0 \rho_m \frac{\partial B}{\partial x} = -\frac{1}{\rho} 1.200\, G \text{ (meter)} = -\frac{1}{\rho} 0.0012\, G \text{ (mm.)}.$$

$$(448)\quad -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho} \rho_m \frac{\partial B}{\partial x} = -\frac{1}{\rho} 0.12236\, G \text{ (meters)} = -\frac{1}{\rho} 0.000\ 12236\, G \text{ (mm.)}.$$

The Evaluation of the Ratios $\frac{dh}{dB}$ *and* $\frac{dh}{dx}$.

We have, from Fig. 17,

$$(449)\quad \frac{dB}{dx} = \frac{B_0 - B}{D_0} = \frac{G}{D_0}.$$

$$(450)\quad \frac{dh}{dx} = \frac{h_0}{D_0} = \frac{h_1}{D_1} = \frac{l}{D_l} \text{ (for the top of the homogeneous atmosphere).}$$

$$(451)\quad \frac{dh}{dB} = \frac{h_0}{B_0 - B} = \frac{D_0}{G}\frac{l}{D_l} = \frac{D_l}{B_l - B}.\frac{l}{D_l} = \frac{l}{B_l} = \frac{7991.04}{0.760} = 10514.5.$$

The pressure $B = 0$ at the top of the homogeneous atmosphere. B_l is the pressure of the atmosphere 0.760 m., and $l = RT$ = the height of the homogeneous atmosphere 7991.04.

For the ratio $\frac{dh}{dx}$, we obtain, by (449) and (451),

$$(452)\quad \frac{dh}{dx} = \frac{dh}{dB}\cdot\frac{B_0 - B}{D_0} = \frac{dh}{dB}\cdot\frac{G}{D_0} = \frac{10514.5}{111\,111} G = 0.09463\, G.$$

This makes it possible to compute approximately the height of the required isobar above the surface, at the horizontal distance from the place of 1 degree = 111 111 meters.

To Find the Difference of Pressure $(B_1 - B)$ *at the Distance* D_1 *that will just balance the Eastward Velocity* v

The eastward velocity of the general circulation v produces a pressure directed southward along the meridian, and it is required to find the difference of pressure in the meridian, higher to the south in the northern hemisphere, that will keep the velocity of motion directed exactly eastward. By equation (414), for steady motion, $\frac{d u}{d t} = 0$, and neglecting the term in w, we have for v in meters per second,

$$(453) \quad \frac{d P}{\rho \, d x} = \cos \theta \, (2 \, \omega_3 + \nu) \, v.$$

Since $d P = g \rho \, d h$, this becomes:

$$(454) \quad g \frac{d h}{d x} = \cos \theta \, (2 \, \omega_3 + \nu) \, v.$$

By equations (452) and (442),

$$(455) \quad g \frac{10514.5}{D_0} \cdot \frac{D_0 (B_1 - B)}{D_1} = \cos \theta \, (2 \, \omega_3 + \nu) \, v. \quad \text{Hence,}$$

$$(456) \quad B_1 - B = \frac{D_1}{10514.5 \, g} \cdot \cos \theta \, (2 \, \omega_3 + \nu) \, v.$$

For any temperature and pressure other than the standard, $T_0 = 273$ and $B_0 = 760$ mm., if we take $D_1 = D_0 = 111\,111\,111$, so that $B_1 = B_0$, that is, $D_1 =$ the pressure at a distance of 1 degree southward on the meridian, we have, for $g = g_0$,

$$(457) \quad B_0 - B = \frac{111\,111\,111}{10514.5 \times 9.806} \cdot \frac{273}{T} \cdot \frac{B}{760} \cdot \cos \theta \, (2 \, \omega_3 + \nu) \, v.$$

The angular velocity of the rotating earth is $2 \, \omega_3 = 0.0001458$ and if ν is neglected, we find,

$$(458) \quad B_0 - B = 0.05644 \frac{B}{T} \, v \cos \theta \text{ (in millimeters).}$$

It should be learned from this example how to apply the general equations of motion on the rotating earth to special cases, by making the proper limitations in the use of the terms.

The Angular Velocity of the Earth's Rotation, ω_3

Allowing for the sidereal time the angular velocity is,

$$(459) \quad \omega_3 = \frac{2 \pi}{\{(23 \times 60) + 56\} 60} = \frac{2 \times 3.14159}{86160} = 0.000072923.$$

(460) $2\,\omega_3 = 0.000145846.$

The Linear, Absolute and Relative Velocities

The absolute linear velocity at any latitude is

(461) $v' = r\,\omega_3 \sin\theta$

at the distance $r = R + h$ from the center of the earth, where $R = 6370191$ meters of 208996600 feet, and h is the height in the atmosphere above the surface. The relative linear velocity of a body moving eastward over the surface of the earth is

(462) $v = r\,\nu \sin\theta.$ Hence,

(463) $$\omega_3 = \frac{v'}{r\sin\theta}, \text{ and } 2\,\omega_3 \cos\theta = \frac{2\,v'\cos\theta}{r\sin\theta} = \frac{2\,v'\cot\theta}{r}.$$

(464) $$\nu = \frac{v}{r\sin\theta}, \text{ and } \nu\cos\theta = \frac{v\cos\theta}{r\sin\theta} = \frac{v\cot\theta}{r}.$$

(465) $$\cos\theta\,(2\,\omega_3 + \nu) = \cos\theta\left(\frac{2\,v'}{r\sin\theta} + \frac{v}{r\sin\theta}\right) = \frac{\cot\theta}{r}(2\,v' + v).$$

It follows that formulas (414) can be written in another form, remembering that $\omega = n$ for the angular velocity,

(466) $$-\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{du}{dt} - \frac{\cot\theta}{r}(2\,v' + v)\,v + \frac{u\,w}{r}.$$

Fourth form of polar equations

$$-\frac{1}{\rho}\frac{\partial P}{\partial y} = \frac{dv}{dt} + \frac{\cot\theta}{r}(2\,v' + v)\,u + (2\,v' + v)\frac{w}{r}.$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{dw}{dt} - \frac{1}{r}(2\,v' + v)\,v - \frac{u^2}{r} + g.$$

In some respects the fourth form of polar equations offers distinct advantages in practical computations, especially by the aid of a few auxiliary tables, such as Tables 104, 105, 106 of the Cloud Report.

Evaluation of the Barometric Gradients in the Fourth Form of the Polar Equations of Motion

From equations (446), (5), and (449), we have:

(467) $$\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{1}{\rho_0}\frac{B_0}{B}\frac{T}{T_0} \cdot g_0\, \rho_m \cdot \frac{G_x}{D_0}.$$

(468) $$\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{760 \times 9.806 \times 13595.8}{1.29305 \times 273 \times 111\,111\,111} \cdot \frac{T}{B} G_x = 0.0025833\,\frac{T}{B} G_x.$$

For steady motion, $\frac{d\,u}{d\,t} = 0$, and neglecting the term in w, in (466)

(469) $$-\,0.0025833\,\frac{T}{B}\,G_x = -\,\frac{\cot\theta}{R}\,(2\,v' + v)\,v.$$ Hence,

(470) $$G_x = 387.1\,\frac{T}{B} \cdot \frac{\cot\theta}{R}\,(2\,v' + v\,)\,v.$$

$$G_y = -\,387.1\,\frac{B}{T} \cdot \frac{\cot\theta}{R}\,(2\,v' + v\,)\,u.$$

$$G_z = 387.1\,\frac{B}{T}\left[\frac{1}{R}\,(2\,v' + v)\,v + \frac{u^2}{R} - g.\right].$$

The gradients are, in millimeters of mercury,

$G_x = (B_1 - B)_x$, along the meridian,

$G_y = (B_1 - B)_y$, along the parallel of latitude,

$G_z = (B_1 - B)_z$ = along the vertical.

As an example of the computation take,

The north polar distance, $\theta \;= 90 - \phi = 30°$,

The radius of the earth, $R = 6370191$, $B = 700$ mm., $T =$ 260° C.

The angular velocity, $2\,\omega_3 \;= 0.00014584$,

The eastward velocity of the earth at θ, $v' = R\,n \sin\theta$,

The relative eastward velocity over the earth, $v = 40$ meters per sec.

	Logarithms		*Logarithms*
$R = 6370191$	6.80415	387.1	2.58782
$2\omega = 0.00014584$	6.16388 − 10	700	2.84510
$\sin\theta = \sin 30°$	9.69897	cot 30°	0.23856
$2v' = 464.5$	2.66700	$(2v' + v)v$	4.30492
$v = 40.0$			9.97640
$2v' + v = 504.5$			
$(2v' + v)v = 2018.0$		T	2.41497
		R	6.80415
$G_x = (B_1 - B)_x = 5.717$		0.75728	9.21912

By Tables 104 and 106	
$\frac{v'}{R}\cot\theta . 2v$	0.005052
$\cot\theta . vv$	0.000435
	0.005487
	Logarithms
$\cot\theta . v\left(\frac{2v'}{R} + v\right)$	7.73933
378.1	2.58782
B	2.84510
	8.17225
T	2.41497
5.717	0.75728

Application of the General Equations of Motion to the Local Circulations in the Earth's Atmosphere

The Local Circulations

The circulations of the earth's atmosphere can be conveniently analyzed under two classes: the first, or *general* circulation, including the large movements that are primarily related to the axis of the earth as the line of reference for the angular motion, and the second, or *local* circulation, including the minor movements that are referred to axes which are wandering over the surface of the earth. The general circulation takes account of the great polar whirls covering the entire hemisphere, one north and one south of the equator, including the trade winds in the tropics, the eastward drifts in the temperate zones, and the minor circulations near the poles. These great zonal currents break up into localized circulations, as determined by the ocean and land areas, which constitute the first disintegration of the general circulation into smaller circulations. The true local circulations are commonly known as cyclones, anticyclones, hurricanes, tornadoes, and waterspouts, and these are referred to axes which move over the earth in paths that are determined by their relation to the breaks in the normal general circulation. Finally, there are very numerous minor whirls, as eddies, small vortices of many types, which constitute the effectual internal friction through the operation of the law of inertia in the moving masses. The analysis of these motions, by the application of the laws of hydrodynamics and thermodynamics given in this chapter, determines the principal problems in theoretical meteorology. It is proposed to set forth the main features of this subject with sufficient fulness to guide other students to the problems of the research which are pressing for solution.

Discussion of the Cylindrical Equations of Motion

The cylindrical equations of motion are most convenient for application to the discussion of the phenomena of the local circulations in cyclones, hurricanes, and tornadoes. There have been several attempts to adapt these equations to the observed data, and the two best-known systems, that of Ferrel and that of the German School of Meteorologists, will be briefly mentioned before taking up the form of vortex that I have been led to adopt in my researches. It will be desirable, at the outset, to assume that the motion is symmetrical about the z-axis in cylindrical co-ordinates, and that the isobars are centered as circles upon this axis, though this is not the case in nature, except for waterspouts, tornadoes, and hurricanes. The pure vortex law does not apply directly to cyclones and anticyclones, and the disturbing terms which make the transition between pure and impure vortices can be studied only by comparing the pure vortex underlying a cyclone with the data obtained by observation. It has been difficult for meteorologists to do this, because the actual conditions in the free air above the surface are found only indirectly by computation, or in an inadequate manner by occasional ascensions with kites and balloons. In recent years, however, enough data have been accumulated to make it possible to advance these studies in the right direction. It will greatly assist those who are engaged in the study of the atmosphere above the ground, if, in planning and executing the observations, the fundamental principles upon which the actual motions must depend are clearly understood.

Ferrel's Local Cyclone

If the vortex is assumed to be symmetrical about the z-axis, and the friction terms $k\,u$, $k\,v$ can be neglected, the second equation in the second form of cylindrical equations (406) becomes:

(471) $\frac{dv}{dt} + (2\,\omega_3 \cos\theta + \nu)\,u = 0.$

Substituting, $\nu = \frac{v}{\varpi}$, $u = \frac{d\varpi}{dt}$, and multiplying by ϖ, this becomes,

(472) $2\,\omega_3 \cos\theta.\ \ \varpi \frac{d\varpi}{dt} + \varpi \frac{dv}{dt} + v \frac{d\varpi}{dt} = 0.$

Integrating, we have for *each* particle in gyration:

(473) $\varpi^2\,\omega_3 \cos\theta + \varpi\,v = \varpi^2\,(\omega_3 \cos\theta + \nu) = c.$

(474) Take $C = \frac{\int c\,dm}{m}$ for the *entire* gyrating mass, if ν_0 is the initial value of ν,

(475) $$C = \frac{\int \varpi^2(\omega_3 \cos\theta + \nu)\,dm}{m} = \tfrac{1}{2}\,\varpi_0^2\,(\omega_3 \cos\theta + \nu_0).$$

If the initial gyration is zero, $\nu_0 = 0$, and we have:

(476) $\varpi^2 \left(\omega_3 \cos\theta + \frac{v}{\varpi}\right) = \tfrac{1}{2}\,\underline{\varpi}_0^2\,\omega_3 \cos\theta.$

This is equal to the moment of inertia of the whole mass at the distance $\frac{1}{2}\,\varpi_0$. We obtain:

(477) $\frac{v}{\varpi} = \frac{\varpi_0^2}{2\,\varpi^2}\,\omega_3 \cos\theta - \omega_3 \cos\theta$, and

$v = \left(\frac{\varpi_0^2}{2\,\varpi^2} - 1\right)\varpi\,\omega_3 \cos\theta$, the tangential velocity at ϖ_0.

If the tangential velocity vanishes, $v = 0$, for $\omega = R$, we have:

(478) $R^2 = \frac{\varpi_0^2}{2}$, and $R = 0.707\,\varpi_0.$

If a cylinder with the radius ϖ_0 is drawn around the gyrating mass, then at the distance 0.707 ϖ_0 the velocity is zero; inside this R the fluid rotates in one direction and outside of it the rotation is in the opposite direction. The general circulation of the atmosphere is arranged to operate in this way, the air north of the latitude 35° 16′ rotating eastward, and the air south of this parallel rotating westward. If water be enclosed in a cylindrical vessel, and an upward current be formed in the center, by heating the lower surface, or by a wheel operating in the water to raise a central column, it will circulate in this way. The water will gyrate about the central axis as it rises, till at the distance 0.707 ϖ_0 from the center the gyration ceases, while beyond this radius the gyration reverses and the water descends in a ring bounded on the outside by the vessel. Ferrel supposed that this type of vortex would represent the local cyclone as well as the general circulation, but this is not the case. In the general circulation there is a fixed mass in gyration, as there is in the cylindrical vessel of the experiment, but observations show that the cyclone, hurricane, and tornado are constructed by means of another type of vortex in which the mass is continuously changing. Other mechanical difficulties are mentioned in the Cloud Report, 1898.

The German Local Cyclone

In the second form of cylindrical equations (406), the second equation becomes:

(479) $$\frac{dv}{dt} + \frac{uv}{\varpi} + \lambda u + kv = 0,$$

where $\lambda = 2\,\omega_3 \cos\theta$ and k is the coefficient of friction. This equation has two solutions, and the vortex has been divided into an inner and an outer part to correspond with them. Thus

		First Solution (*Inner Part*)	*Second Solution* (*Outer Part*)
(480)	Radial velocity	$u = -\frac{c}{2}\varpi.$	$u = -\frac{c}{\varpi}.$
(481)	Tangential velocity	$v = \frac{\lambda}{k-c}\cdot\frac{c}{2}\varpi.$	$v = \frac{\lambda}{k}\,\frac{c}{\varpi}.$
(482)	Vertical velocity	$w = cz.$	$w = 0.$

The constant c depends upon the dimensions of the vortex, and must be determined by observations. These values of u, v, w, can be readily verified by substituting in equation (479).

If we take the following current functions, called the Stokes functions, and the vortex law,

(483) $$u = -\frac{1}{\varpi}\frac{\partial \psi}{\partial z}$$

(484) $$w = +\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}$$

(485) $$v\,\varpi = \psi$$

there results for the two solutions,

		First Solution (*Inner Part*)	*Second Solution* (*Outer Part*)
(486)	Current Function.	$\psi_1 = \frac{c}{2}\varpi^2 z$	$\psi_1 = + c\,z.$
(487)	Radial Velocity.	$u = -\frac{1}{\varpi}\frac{\partial \psi_1}{\partial z} = -\frac{c}{2}\varpi.$	$u = -\frac{1}{\varpi}\frac{\partial \psi_1}{\partial z} = -\frac{c}{\varpi}.$
(488)	Vertical Velocity.	$w = +\frac{1}{\varpi}\frac{\partial \psi_1}{\partial \bar{\omega}} = + c\,z.$	$w = +\frac{1}{\varpi}\frac{\partial \psi_1}{\partial z} = 0.$
(489)	Vortex Law.	$v\,\varpi = \psi_2 = \frac{\lambda}{k-c}\frac{c}{2}\varpi^2.$	$v\,\varpi = \psi_2 = \frac{\lambda}{k}c.$
(490)		$\psi_2 = \frac{\lambda}{k-c}\frac{\psi_1}{z}.$	$\psi_2 = \frac{\lambda}{k}\frac{\psi_1}{z}.$

It is seen that two forms of the current function are required to satisfy the general vortex law which will be deduced later. Even if the constants k and c could be determined the solution is not consistent for the entire vortex in either the inner or the outer part. There is in nature no such division of the vortex, that is, there is no outward part without vertical velocity, as compared with an inward part having vertical velocity, $w = + c\,z$. It is for this reason that the application of these formulas to the cyclone has not been successful. The cyclone is constructed upon quite different principles. The solutions of the second equation of motion can be satisfied by yet other values, which give a consistent current function for all values of the currents and the vortex law.

The General Equation of Cylindrical Vortices

If the definition of a cylinder be extended to include any figure of revolution formed by rotating any line as a generatrix about an axis, the vortices of meteorology can be designated as cylindrical or columnar vortices, the axis being approximately vertical in direction. The cylindrical equations of motion will therefore be adopted, and they are transformed in the following manner. In discussing problems in vortex motion, it is convenient to use the current function ψ, which is deduced from the equation of continuity (394):

(492) $$\frac{\partial u}{\partial \varpi} + \frac{u}{\varpi} + \frac{\partial w}{\partial z} = 0.$$

This may be put into another form,

(493) $$\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(u\varpi) + \frac{\partial w}{\partial z} = 0.$$

This is satisfied by substituting the velocities,

(494) $$u = -\frac{1}{\varpi}\frac{\partial \psi}{\partial z},$$

(495) $$w = +\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi},$$

which are known as Stokes's functions.

In order that the equation of continuity may satisfy the second equation of motion, assuming steady motion and $\frac{\partial v}{\partial t} = 0$, this becomes, from $(401)_2$, with no deflecting force and no friction,

(496) $$u\frac{\partial v}{\partial \varpi} + w\frac{\partial v}{\partial z} + \frac{u\,v}{\varpi} = 0,$$

and it is sufficient to make

(497) $$v = \frac{\psi}{\varpi},$$

so that $v \varpi = \psi =$ constant is the usual vortex law, or in its most general form, $v \varpi = a \,.\, \psi$, where the ψ of the Stokes functions is made to cover the vortex law by the constant factor a. By differentiation,

(498) $$\frac{\partial v}{\partial \varpi} = \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} - \frac{\psi}{\varpi^2},$$

(499) $$\frac{\partial v}{\partial z} = \frac{1}{\varpi}\frac{\partial \psi}{\partial z}.$$

Substituting these values in (496), it becomes:

(500) $$-\frac{1}{\varpi^2}\frac{\partial \psi}{\partial \varpi}\frac{\partial \psi}{\partial z} + \frac{1}{\varpi^3}\frac{\partial \psi}{\partial z}\psi + \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}\frac{\partial \psi}{\partial z} - \frac{1}{\varpi^3}\frac{\partial \psi}{\partial z}\psi = 0.$$

If the equation (496) is multiplied by ϖ, since $\frac{\partial \varpi}{\partial z} = 0$, it can be written,

(501) $$u v + u \varpi \frac{\partial v}{\partial \varpi} + w \varpi \frac{\partial v}{\partial z} + w v \frac{\partial \varpi}{\partial z} = 0,$$ and, therefore, in the form,

(502) $$u \frac{\partial}{\partial \varpi}(\varpi v) + w \frac{\partial}{\partial z}(\varpi v) = 0.$$

This shows that $\varpi v = \psi =$ constant is a solution of the equation of continuity. Any function of ψ which satisfies this equation will be a solution of the second equation of motion. Inasmuch as there are several such values of ψ known, it is only necessary to choose the one which is in harmony with the observed phenomena in the earth's atmosphere in order to obtain the solution of the motions found in cyclones, hurricanes, and tornadoes, or waterspouts. Hence an arbitrary function of ψ,

(503) $\varpi v = f(\psi)$,

is a solution of the second equation of motion.

The potential and the pressure terms in the first and third equations can be eliminated by the following process. From (401) in the case of symmetry about the z-axis,

(504) $$-\frac{\partial V}{\partial \varpi} - \frac{\partial P}{\rho \partial \varpi} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \varpi} + w\frac{\partial u}{\partial z} - \frac{v^2}{\varpi}.$$
$$-\frac{\partial V}{\partial z} - \frac{\partial P}{\rho \partial z} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial \varpi} + w\frac{\partial w}{\partial z}.$$

Differentiate the first equation to ∂z, the second to $\partial \varpi$, and subtract with the result,

$$(505)\quad 0 = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial \varpi}\right) + \frac{\partial u}{\partial \varpi}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial \varpi}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial \varpi}\right) - \frac{\partial}{\partial z}\left(\frac{v^2}{\varpi}\right).$$

We derive the following auxiliary differentiations:

$$(506)\quad \frac{\partial u}{\partial \varpi} = -\frac{\partial \psi}{\partial z}\frac{\partial}{\partial \varpi}\frac{1}{\varpi}. \qquad \frac{\partial u}{\partial z} = -\frac{1}{\varpi}\frac{\partial^2 \psi}{\partial z^2}.$$

$$(507)\quad -\frac{\partial w}{\partial \varpi} = -\frac{1}{\varpi}\frac{\partial^2 \psi}{\partial \varpi^2} + \frac{1}{\varpi^2}\frac{\partial \psi}{\partial \varpi}. \qquad \frac{\partial w}{\partial z} = \frac{\partial \psi}{\partial \varpi}\frac{\partial}{\delta z}\frac{1}{\varpi}.$$

$$(508)\quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial \varpi} = -\frac{1}{\varpi}\left(\frac{\partial^2 \psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} + \frac{\partial^2 \psi}{\partial z^2}\right).$$

Since $v\,\varpi = f(\psi)$, $v^2\,\varpi^2 = [f(\psi)]^2$, $\frac{v^2}{\varpi} = \frac{[f(\psi)]^2}{\varpi^3}$, we have,

$$(509)\quad \frac{\partial}{\partial z}\frac{v^2}{\varpi} = \frac{2f(\psi)\frac{\partial f(\psi)}{\partial z}}{\varpi^3}.$$

Making these substitutions in (505), we obtain,

$$(510)\quad 0 = \frac{1}{\varpi}\frac{\partial}{\partial t}\left(\frac{\partial^2 \psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} + \frac{\partial^2 \psi}{\partial z^2}\right).$$
$$+\frac{\partial \psi}{\partial z}\frac{\partial}{\partial \varpi}\left[\frac{1}{\varpi^2}\left(\frac{\partial^2 \psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} + \frac{\partial^2 \psi}{\partial z^2}\right)\right]$$
$$-\frac{\partial \psi}{\partial \varpi}\frac{\partial}{\partial z}\left[\frac{1}{\varpi^2}\left(\frac{\partial^2 \psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} + \frac{\partial^2 \psi}{\partial z^2}\right)\right] - \frac{2f(\psi)\frac{\partial f(\psi)}{\partial z}}{\varpi^3}.$$

Any function of ψ satisfying this equation is capable of giving a vortex motion. In the application to the atmosphere some simple forms will be considered and illustrated by examples. The first form,

$$(511)\quad \psi = A\,\varpi^2 z,$$

gives a funnel-shaped vortex, and the second form,

$$(512)\quad \psi = A\,\varpi^2 \sin a z,$$

gives a dumb-bell-shaped vortex. These are the common ones in the atmosphere, as will be shown by the observations. Unfortunately the motions under the complex local forces that generate storms do not often produce pure vortex motion, but it is the province of meteorology to consider the perturbations as observed and to give an account of their causes.

The Angular Velocity

By formula (371), the angular velocity is,

$$(513)\quad 2\,\omega_z = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial \varpi},$$

and if an arbitrary function of ψ is taken,

$$(514)\quad f(\psi) = \frac{1}{\varpi^2}\left(\frac{\partial^2 \psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} + \frac{\partial^2 \psi}{\partial z^2}\right),$$

it follows that,

$$(515)\quad 2\,\omega = \varpi f(\psi) = \frac{1}{\varpi}\left(\frac{\partial^2 \psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} + \frac{\partial^2 \psi}{\partial z^2}\right).$$

By differentiations it follows that

$$(516)\quad \frac{\partial}{\partial \varpi} f(\psi) = \frac{\partial f(\psi)}{\partial \psi} \cdot \frac{\partial \psi}{\partial \varpi}$$

$$(517)\quad \frac{\partial}{\partial z} f(\psi) = \frac{\partial f(\psi)}{d\psi} \cdot \frac{\partial \psi}{\partial z}.$$

The Total Pressure

Since by formula (418), omitting Q and J,

$$(518)\quad -\frac{P}{\rho} = \tfrac{1}{2}\,(u^2 + v^2 + w^2) + g\,z,$$

it follows, by using Stokes's functions, that,

$$(519)\quad -\frac{P}{\rho} = \frac{1}{2\varpi^2}\left[\left(\frac{\partial \psi}{\partial \varpi}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2\right] + \tfrac{1}{2}\,\frac{a^2 \psi^2}{\varpi^2} + g\,z,$$

for one limit. The difference of pressure between two points, designated by n and $n+1$, becomes, by using the mean density ρ_m,

$$(520)\quad P_{n+1} - P_n = \frac{\rho_m}{2}\left\{\frac{1}{\varpi^2{}_n}\left[\left(\frac{\partial\psi}{\partial\varpi}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right]_n - \frac{1}{\varpi^2_{n+1}}\left[\left(\frac{\partial\psi}{\partial\varpi}\right)^2 + \left(\frac{\partial\psi}{\partial\varpi}\right)^2\right]_{n+1}\right\} + \frac{\rho_m}{2}\left\{\left(\frac{a^2\psi^2}{\varpi^2}\right)_n - \left(\frac{a^2\psi^2}{\varpi^2}\right)_{n+1}\right\} + \rho_m g_m (z_n - z_{n+1}).$$

The Application of the Vortex Formulas to the Funnel-shaped Tube

Employing formulas (494), (495), (497), (503), we readily obtain the following group of relations,

$$(521)\quad \psi = C\varpi^2 z = v\varpi = u\varpi z = -\frac{w}{2}\varpi^2. \qquad \text{Current function.}$$

$$(522)\quad C = \frac{\psi}{\varpi^2 z} = \frac{v}{\varpi z} = \frac{u}{\varpi} = -\frac{w}{2z}. \qquad \text{Vortex constant.}$$

$$(523)\quad u = +\frac{1}{\varpi}\frac{\partial\psi}{\partial z} = C\varpi = \frac{\psi}{\varpi z} = -\frac{w\varpi}{2z} = \frac{v}{z}. \qquad \text{Radial velocity.}$$

$$(524)\quad v = \frac{\psi}{\varpi} = C\varpi z = \frac{\psi}{\varpi} = -\frac{w}{2}\varpi = uz. \qquad \text{Tangential velocity.}$$

$$(525)\quad w = -\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} = -2Cz = -\frac{2\psi}{\varpi^2} = -\frac{2v}{\varpi} = -\frac{2uz}{\varpi}.$$

Vertical velocity.

The Application of the Vortex Formulas to the Dumb-Bell-Shaped Tube

$$(526)\quad \psi = A\varpi^2 \sin az.$$

$$(527)\quad u = -\frac{1}{\varpi}\frac{\partial\psi}{\partial z} = -\frac{A a \varpi^2 \cos az}{\varpi} = -A a \varpi \cos az.$$

$$(528)\quad v = \frac{a\psi}{\varpi} = +\frac{A a \varpi^2 \sin az}{\varpi} = +A a \varpi \sin az.$$

$$(529)\quad w = +\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} = +\frac{2A\varpi \sin az}{\varpi} = +2A \sin az.$$

The Total Pressure

For the funnel-shaped tube, omitting the expansion terms,

(530) $$-\frac{P}{\rho} = \frac{1}{2}(u^2 + v^2 + w^2) + g z = \frac{1}{2} C^2 (\varpi^2 + \varpi^2 z^2 + 4 z^2) + g z + \text{constant.}$$

For the dumb-bell-shaped tube,

(531) $$-\frac{P}{\rho} = \frac{1}{2}(A^2 a^2 \varpi^2 \cos^2 az + A^2 a^2 \varpi^2 \sin^2 az + 4 A^2 \sin^2 az) + g z + \text{constant.}$$

(532) $$-\frac{P}{\rho} = \frac{1}{2} A^2 a^2 \varpi^2 + 2 A^2 \sin^2 az + g z + \text{constant.}$$

(533) $$-\frac{P}{\rho} = \frac{1}{2} A^2 a^2 \varpi^2 + A^2 (1 - \cos 2\, az) + g z + \text{constant.}$$

It should be noted that the signs of the Stokes functions have been taken opposite to one another in the funnel-shaped and the dumb-bell-shaped vortices. This is because it is more convenient to place the plane of reference for the funnel-shaped vortex at the base of the cloud from which it is developed, with the positive direction of the z-axis downward, while in the dumb-bell-shaped vortex the first plane of reference is taken at or below the surface of the sea or ground, and the positive direction of the z-axis is upward to the second plane of reference. These will be explained further by diagrams and examples.

The Relations Between Successive Vortex Tubes

A vortex is so constructed that a section through it perpendicular to the z-axis at any height, z or az, cuts off a series of rings so regulated in size that the successive radii stand in a constant ratio to each other. Take this ratio,

(534) $$\rho = \frac{\varpi_n}{\varpi_{n+1}}, \quad \text{and}$$

(535) $$\log \rho = \log \frac{\varpi_n}{\varpi_{n+1}}.$$

If ϖ_n is the radius of the outer ring ϖ_1, and ϖ_2, ϖ_3, ϖ_4 ... of the successive rings inward, then,

$$(536)\quad \rho = \frac{\varpi_1}{\varpi_2} = \frac{\varpi_2}{\varpi_3} = \frac{\varpi_3}{\varpi_4} \ldots\ldots$$

This constant ratio ρ plays a very important part in the computation of these vortices, and it is found that we can pass from one value of the radius and the velocities to the w others in succession by employing the following formulas:

(535) Ratio of the radii. $\log \rho = \log \dfrac{\varpi_n}{\varpi_{n+1}}$.

(536) Vortex constant, $\log C_n = \log C_1 + 2\,n \log \rho$.

(537) Radii of rings, $\log \varpi_n = \log \varpi_1 - n \log \rho$.

(538) Radial velocity, $\log u_n = \log u_1 + n \log \rho$.

(539) Tangential velocity, $\log v_n = \log u_1 + n \log \rho$.

(540) Vertical velocity, $\log w_n = \log w_1 + 2\,n \log \rho$.

(541) Horizontal angle i, $\log \tan i =$ constant.

(542) Vertical angle η, $\log \tan \eta = \log \tan \eta_1 + n \log \rho$.

(543) Time of one rotation t, $\log t_n = \log t_1 - 2\,n \log \rho$.

(544) Volume through rings, $V = \pi\,({}^2\varpi_n - \varpi^2_{n+1})\,w_m =$ constant.

(545) Centrifugal force, $\log \left(\dfrac{v^2}{\varpi}\right)_n = \log \left(\dfrac{v^2}{\varpi}\right)_1 + 3\,n \log \rho$.

(546) Barometric pressure, $\log \dfrac{B_n - B_{n+1}}{B_{n-1} - B_n} = \log \dfrac{\varpi_n}{\varpi_{n+1}} + \log \rho$.

(547) Total velocity q, $q = (u^2 + v^2 + w^2)^{\frac{1}{2}} = v \sec i \sec w$.

The relations shown by these formulas will be made clearer by a diagram giving the connection between the angles and the velocities.

(548) $\tan i = \dfrac{u}{v}$. (549) $\varpi = \left(\dfrac{a\,\psi}{Aa \sin az}\right)^{\frac{1}{2}}$.

(550) $\tan \eta = \dfrac{w}{v \sec i}$.

$\varpi, \phi =$ the cylindrical co-ordinates of a point on the $x\,y$ plane
$q\,(u, v, w) =$ the co-ordinates of the velocity at the point (ϖ, ϕ, z).
$\sigma\,(u, v) =$ the component of q on the horizontal plane; $i =$ the

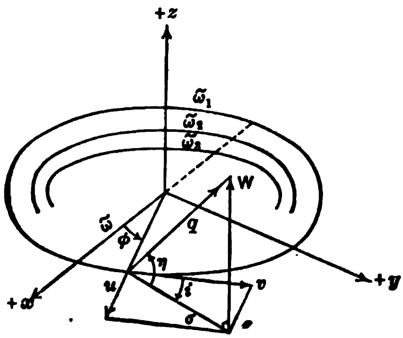


FIG. 18. The relations of the angles and velocities in the formulas.

angle from v to σ, positive outward from the tangent; $\eta =$ the angle from σ to q, positive upwards.

The Second Form of the Cylindrical Equations of Motion (406) *in Terms of the Current Function* ψ

The equations of motion for cylindrical vortices can be readily transformed into terms depending upon the current function ψ. Writing equations (406) in their full forms, they become,

$$(552)\quad -\frac{1}{\rho}\frac{\partial P}{\partial \varpi} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \varpi} + w\frac{\partial u}{\partial z} - \frac{v^2}{\varpi} - 2\,\omega_3 \cos\theta \,.\, v + k\,u.$$

$$-\frac{1}{\rho}\frac{\partial P}{\varpi \partial \phi} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial \varpi} + w\frac{\partial v}{\partial z} + \frac{u v}{\varpi} + 2\,\omega_3 \cos\theta \,.\, u + k\,v.$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial \varpi} + w\frac{\partial w}{\partial z} + g + k\,w.$$

It is only necessary to develop the differential terms from the velocities given in (523)–(525) and (527)–529).

For the Funnel-Shaped Vortex

$$(553)\quad \begin{cases} u = C\,\varpi. & v = C\,\varpi\,z. & w = -\,2\,C\,z. \\ \dfrac{\partial u}{\partial \varpi} = C. & \dfrac{\partial v}{\partial \varpi} = C\,z. & \dfrac{\partial w}{\partial \varpi} = 0 \\ \dfrac{\partial u}{\partial z} = 0. & \dfrac{\partial v}{\partial z} = C\,\varpi. & \dfrac{\partial w}{\partial z} = -\,2\,C. \end{cases}$$

$$(554)\quad \begin{cases} u\dfrac{\partial u}{\partial \varpi} = C^2\varpi. & w\dfrac{\partial u}{\partial z} = 0. & \dfrac{v^2}{\varpi} = C^2\varpi z^2. \\ u\dfrac{\partial v}{\partial \varpi} = C^2\varpi z. & w\dfrac{\partial v}{\partial z} = -2C^2\varpi z. & \dfrac{uv}{\varpi} = C^2\varpi z. \\ u\dfrac{\partial w}{\partial \varpi} = 0. & w\dfrac{\partial w}{\partial z} = 4C^2 z. & \end{cases}$$

Hence the second form of the cylindrical equations of motion becomes, for the funnel-shaped vortex,

$$(555)\quad \begin{aligned} -\frac{1}{\rho}\frac{\partial P}{\partial \varpi} &= \frac{\partial u}{\partial t} + C^2\varpi - C^2 z^2\varpi - 2\varpi_3\cos\theta\,.\,v + k\,u. \\ -\frac{1}{\rho\varpi}\frac{\partial P}{\partial \phi} &= \frac{\partial v}{\partial t} + \quad 0 \quad + 2\varpi_3\cos\theta\,.\,u + k\,v. \\ -\frac{1}{\rho}\frac{\partial P}{\partial z} &= \frac{\partial w}{\partial t} + \quad 4C^2 z + g \quad + k\,w. \end{aligned}$$

Multiply the equations (555), respectively, by $\partial\varpi$, $\varpi\,\partial\phi$, and ∂z, and integrate for the total pressure, and we obtain, omitting the friction terms,

$$(556)\quad -\int\frac{\partial P}{\rho} = -\frac{P}{\rho} = \tfrac{1}{2}(u^2+v^2+w^2) + \tfrac{1}{2}C^2\varpi^2 - \tfrac{1}{2}C^2z^2\varpi^2 + 2C^2z^2 + g\,z.$$

Substituting the values of $u^2{}_1$, $v^2{}_1$, $w^2{}_1$ this becomes:

$$(557)\quad -\frac{P}{\rho} = C^2\varpi,^2 + 4C^2z^2 + g\,z, \text{ at the point } (\varpi, z).$$

The difference of pressure between two points $(\varpi_1\,z)_n$ and $(\varpi_1\,z)_{n+1}$ may be expressed,

$$\frac{P_n - P_{n+1}}{\rho_m} = (C^2\varpi^2)_{n+1} - (C^2\varpi^2)_n + 4\left[(C^2z^2)_{n+1} - (C^2z^2)_2\right] + g\,(z_{n+1} - z_n).$$

It has been customary in meteorology to use the formula (530) as an expression for the total pressure integral, but it is evident that (556) and (557) are the complete forms for the funnel-shaped vortex. If the inertia terms are omitted the formula becomes, without friction,

$$(558)\quad -\frac{P}{\rho} = \tfrac{1}{2}C^2\varpi^2 - \tfrac{1}{2}C^2z^2\varpi^2 + 2C^2z^2 + g\,z.$$

Hence, we can summarize the result for $-\frac{P}{\rho}$:
(556) includes the inertia and the expansion terms; (519) contains the inertia, but omits the expansion; (558) omits the inertia but contains the expansion.

For the Dumb-Bell-Shaped Vortex

From the equations (527)–(529) we have by differentiation and substitution, using $\psi = A\,\varpi^2 \sin az$,

$$(559)\quad \begin{cases} u = -A\,a\,\varpi \cos az. \\ \dfrac{\partial u}{\partial \varpi} = -A\,a \cos az. \\ \dfrac{\partial u}{\partial z} = A\,a^2\,\varpi \sin az. \end{cases} \qquad \begin{cases} v = A\,a\,\varpi \sin az. \\ \dfrac{\partial v}{\partial \varpi} = A\,a \sin az. \\ \dfrac{\partial v}{\partial z} = A\,a^2\varpi \cos az. \end{cases}$$

$$\begin{cases} w = z\,A \sin az. \\ \dfrac{\partial w}{\partial \varpi} = 0. \\ \dfrac{\partial w}{\partial z} = 2\,A\,a \cos az. \end{cases}$$

$$(560)\quad \begin{cases} u\dfrac{\partial}{\partial \varpi} = A^2\,a^2\,\varpi \cos^2 az. \\ u\dfrac{\partial v}{\partial \varpi} = -A^2a^2\varpi \sin az \cos az. \\ u\dfrac{\partial w}{\partial z} = 0. \end{cases} \qquad \begin{cases} w\dfrac{\partial u}{\partial z} = 2\,A^2\,a^2\,\varpi \sin^2 az. \\ w\dfrac{\partial v}{\partial z} = 2A^2a^2\varpi \sin az \cos az. \\ w\dfrac{\partial w}{\partial z} = 4A^2a \sin az \cos az. \end{cases}$$

$$\begin{cases} -\dfrac{v^2}{\varpi} = -A^2\,a^2\,\varpi \sin^2 az. \\ \dfrac{uv}{\varpi} = -A^2\,a^2\,\varpi \sin az \cos az. \\ \dfrac{dv}{dz} = g. \end{cases}$$

Substitute these values in the general equations (552):

$$(561)\quad \begin{cases} -\dfrac{1}{\rho}\dfrac{\partial P}{\partial \varpi} = \dfrac{\partial u}{\partial t} + A^2 a^2 \varpi - 2\,\varpi_3 \cos\theta \,.\, v + k\,u. \\ -\dfrac{1}{\rho}\dfrac{\partial P}{\varpi\,\partial \phi} = \dfrac{\partial v}{\partial t} + 0 + 2\,\varpi_3 \cos\theta \,.\, u + k\,v. \\ -\dfrac{1}{\rho}\dfrac{\partial P}{\partial z} = \dfrac{\partial w}{\partial t} + 2\,A^2 .\, 2 \sin a\,z \cos a\,z \,.\, a + g + k\,w. \end{cases}$$

Multiply by $\partial \varpi$, $\varpi \partial \phi$, ∂z, respectively, omit the k-terms, and integrate for the total pressure, remembering that,

(562) $2\int \sin az \cos a \, \partial \, az = \sin^2 az,$

(563) $-\frac{P}{\rho} = \frac{1}{2}(u^2 + v^2 + w^2) + \frac{1}{2} A^2 a^2 \varpi^2 + 2 A^2 \sin^2 az + g z.$

The term for the velocity square is,

(564) $\frac{1}{2}(u^2 + v^2 + w^2) = \frac{1}{2} A^2 a^2 \varpi^2 + 2 A^2 \sin^2 az,$

so that the total pressure with the inertia and expansion becomes:

(565) $-\frac{P}{\rho} = A^2 a^2 \varpi^2 + 4 A^2 \sin^2 az + g z$, for the point $(\varpi \, z)$.

The total pressure without the expansion is:

(566) $-\frac{P}{\rho} = \frac{1}{2} A^2 a^2 \varpi^2 + 2 A^2 \sin^2 az + g z.$

The total pressure without the inertia is:

(567) $-\frac{P}{\rho} = \frac{1}{2} A^2 a^2 \varpi^2 + 2 A^2 \sin^2 az + g z.$

It is apparent that in the dumb-bell-shaped vortex the pressure difference $\frac{P_n - P_{n+1}}{\rho_m}$, required to overcome the inertia resistance, is the same as that which is needed to overcome the resistance to expansion.

The Deflecting Force

The terms in θ, the polar distance, $-2\,\omega_3 \cos\theta \,.\, v$ and $+2\,\omega_3 \cos\theta \,.\, u$, disappear from the equation of total pressure in the summation because we have,

(568) $v \, d\varpi = u \, \varpi \, d\phi,$

just as in the rectangular co-ordinates,

(569) $v \, dx = u \, dy,$

which shows that the deflecting force is at right angles to the direction of motion. The centrifugal force $\frac{v^2}{\varpi}$, $\frac{u v}{\varpi}$, being at right angles to the direction of motion and induced by the velocities u, v, together with the deflecting force, has no influence upon the circulation except to change the direction without producing acceleration. In the same way a planet falls toward a body exerting central force, and thus moves in an orbit about it, but the velocity in the orbit is not changed by these forces acting at right angles to the direction of motion.

The Force of Friction

The viscous friction in the atmosphere is a very small quantity, and k would be a small coefficient were it not that in all large movements of the air there are numerous small vortices produced within and carried along in the great current. These minor whirls have a strong force of resistance and they are largely concerned in frittering down the energy of motion contained in the large current. It is customary to take the term expressing friction proportional to the velocity, $k\,u$, $k\,v$, $k\,w$.

This is a subject that has not been satisfactorily cleared up, and it will require much careful research. There can be no doubt that k is a variable coefficient, and differs widely in tornadoes passing over a city or rough country from that in a cyclone over an ocean area. It is not certain that the velocity enters the equations as the simple first power, but that remains to be determined. The resistance due to the friction, whatever function may be found to express it, acts along the line of motion to retard the velocity, so that the pressure difference must increase to overcome this type of resistance.

We may write the final equation for all the terms, when the pressure-difference between two points is required, using $(\varpi, z)_{n+1}$ and $(\varpi, z)_n$,

$$(570)\quad -\frac{P}{\rho_m}\Big]_n^{n+1} = A^2 a^2 \varpi^2\Big]_n^{n+1} + 4\,A^2 \sin^2 a\,z\Big]_n^{n+1} + g\,(z_{n+1} - z_n) + k\,q\Big]_n^{n+1}.$$

The mean density ρ_m along the path between the two points must be used, and in general the mean conditions of all the terms along the path of the integration must be carefully considered.

The Transformation of Energy in the Circulation of the Atmosphere

The circulation of the atmosphere is the process of the transformation of energy, the transportation of warm and cold masses of air from one place to another being the evidence that a disturbed thermal condition is seeking its normal equilibrium. These currents are so complex that at present there is no possibility of working out a comprehensive system of equilibrium. The direction and velocity of the currents in all levels and in all latitudes and the temperatures of the masses, must be worked out by numerous observations before that can be undertaken. All the integrations heretofore proposed assume that a nearly perfect vortex law can be laid at the base of the discussion of the general and the local circulations, but as the vortices on the hemisphere and in the cyclones are very imperfect a more complicated treatment is necessary. At present it is possible to lay down only some isolated, detached propositions which contribute to the ultimate solution of the problems of atmospheric circulation. The following discussions merely introduce a subject of great value, which is capable of unlimited development.

CASE I. *The Change of Position of the Layers in a Column of Air*

When a layer of air in a column is not at the temperature which belongs to its elevation it must move upward or downward in order to gain a position of equilibrium, upward if too warm, and downward if too cold for its place. This occurs when a cold sheet overruns a warm layer, when there will be an interchange of position in certain streams, which may have a vortical structure more or less fully developed. The following propositions take no account of the form of the current lines, but they explain the amount of energy that can be transformed into a velocity q. The chief imperfection in these propositions consists in the omission of the powerful heat terms $(Q_1 - Q_0)$.

From the equations (196) to (199), we find,

$$(571) \quad -\frac{P_1-P_0}{\rho_{10}} = g\ (z_1-z_0)+\tfrac{1}{2}(q_1^2-q_0^2)+(Q_1-Q) = -n_1 Cp_a (T_1-T_0)+n_1\ (Cp_a-Cp_{10})(T_1-T_0)$$

so that the velocity equation becomes, for the mass M,

$$(572) \quad \tfrac{1}{2}(q_1^2-q_0^2)\,M = -g\,(z_1-z_0)\,M-(n_1-n_0)\,Cp_{10}\,(T_1-T_0)\,M$$
$$M - (Q_1 - Q_0)\,M.$$

The evaluation of the term $-(n_1 - n_0)\,Cp_{10}\,(T_1 - T_0)\,m$ is difficult, because the moment a mass of air moves up or down, it at the first instant has an adiabatic gradient, $n_0 = 1$, of expansion or contraction, which sets up a minor circulation within the mass whose gradient is n_1, so that this internal circulation cannot be followed, and it is necessary to treat it as a resultant mass whose general gradient is n_1. We, therefore, omit this

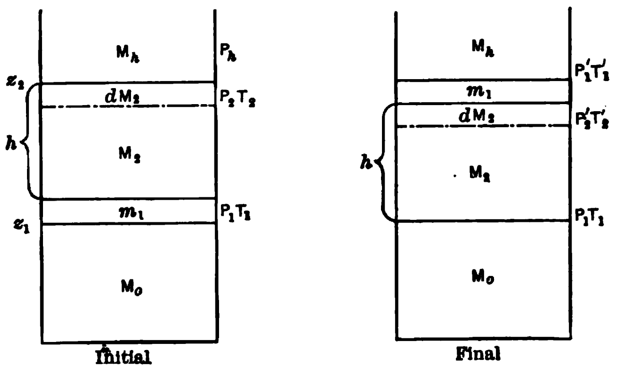


FIG. 19. Change of position of the layers in a column of air.

term, also the initial velocity q_0^2 and the initial height z_0 for convenience, and have for the kinetic energy, for several masses,

$$(573) \quad \tfrac{1}{2}\,m\,q^2 = \Sigma\,[-n_1\,Cp_{10}\,(T_1 - T_0)\,m - g\,z\,m].$$

These terms must be evaluated and substituted in the general formula. (Compare Margules' "Energie der Stürme.")

Change of Position of the Layers in a Column of Air

Suppose that the thin layer m_1 at the height z_1, pressure P_1, temperature T_1, is too warm for its place, but that it must rise to the height z_2 to be in equilibrium, while a column M_2 of height h falls through a short distance. The mass M_0 is not affected while the mass M_h above z_2 falls as in a piston without changing the pressure or temperature. The changes in the mass M_2 must be integrated through the layers, $M_2 = \int_0^h d\,M_2$. In the exchanges the pressure of m_1 changes from P_1 to P_1', and the temperature from T_1 to T_1'; the pressure of the differential layer $d\,M_2$ changes from P_2 to P_2', and the temperature from T_2 to T_2', while the height h changes slightly as the large mass contracts in falling.

	Layer	*Initial*	*Final*
(574)	$d\,M_2$	$P_2\,T_2$	$P_2' = P_2 + g\,m_1.\quad T_2' = T_2\left(\frac{P_2'}{P_2}\right)^{\frac{k-1}{n\,k}}.$
	m_1	$P_1\,T_1$	$P_1' = P_h + g\,M_2.\quad T_1' = T_1\left(\frac{P_1'}{P_1}\right)^{\frac{k-1}{n\,k}}.$

To evaluate T_2' we have

(575) $$T_2' = T_2\left(\frac{P_2'}{P_2}\right)^{\frac{k-1}{n\,k}} = T_2\left(1 + \frac{g\,m_1}{P_2}\right)^{\frac{k-1}{n\,k}} = T_2\left(1 + \frac{k-1}{n\,k}\frac{g\,m_1}{P_2}\right) = T_2 + T_2\frac{R}{n\,Cp}\frac{g\,m_1}{P_2}.$$

Substitute these values in equation (573).

(576) $$\tfrac{1}{2}\,m_1\,q^2 = n_1\,Cp_{10}\left[\int (T_1 - T_1')\,d\,m_1 + \int (T_2 - T_2')\,d\,M_2\right] - g\int m_1\,d\,z - g\int d\,M_2\,d\,z.$$

(577) $$\tfrac{1}{2}\,m_1\,q^2 = n_1\,Cp_{10}\left[T_1 - T_1\left(\frac{P_1'}{P_1}\right)^{\frac{k-1}{n\,k}}\right] m_1 - g\,(z - h)\,m_1,$$

since in the $d\,M_2$-term,

(578) $$n_1\,Cp_{10}\left(T_2 - T_2 - \int \frac{T_2}{P_2}\frac{R}{n_1\,Cp_{10}}\,g\,m_1\,d\,M_2\right) = - g\,m_1 \int \frac{dM_2}{\rho_2} = - g\,m_1 \int d\,z = -g\,m_1\,h.$$

The two gravity terms in (576) nearly disappear by the summation. The available kinetic energy $\frac{1}{2}\,m_1\,q^2$ caused by displacing a thin layer by a thick layer can be computed in this way, but there is no account given of the form of the currents produced by the transformation, nor of the energy lost in the small internal vortices with the accompanying inertia and friction, nor the energy lost by radiation.

The Evaluation of $\int T\,d\,m$ in Linear Vertical Temperature Changes

Since the integration of the term $\int T\,d\,m$ may frequently occur for a simple linear vertical gradient, it is proper to secure the general auxiliary theorem that will express this term when the temperature is defined by

(579) $T = T_0 - a\,z.$

If a is not constant, as is seldom the case except for short vertical distances, then another solution will be required.

We have to evaluate, for $T = T_0 - a\,z$,

(580) $\int T\,d\,m = \int T\,\rho\,d\,z.$

It is convenient to have before us the equivalents,

(581) $$\frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{\frac{n\,Cp}{R}} = \left(\frac{T}{T_0}\right)^{\frac{n\,k}{k-1}} = \left(\frac{T}{T_0}\right)^{\frac{-g}{R}\frac{dz}{d\,T}} = \left(\frac{T}{T_0}\right)^{g/Ra}.$$

By substitutions, we find,

(582) $$\int_0^z T\,\rho\,d\,z = \frac{1}{R}\int_0^z P\,d\,z = \frac{1}{R}\int_0^z P_0\left(\frac{T}{T_0}\right)^{g/Ra} d\,z = \frac{1}{R}\int_0^z P_0\,T_0^{-g/Ra}\,T^{g/Ra}\,d\,z.$$

Change the limits of integration from z to T. Since,

(583) $T = T_0 - a\,z,\quad d\,T = -\,a\,d\,z,\quad -\dfrac{dT}{a} = d\,z$, we have:

(584) $$\int_0^z T\,\rho\,d\,z = \frac{1}{Ra}P_0\,T_0^{-g/Ra}\int_T^{T_0} T^{g/Ra}\,d\,T = \frac{1}{Ra}P_0\,T_0^{-g/Ra}\,\frac{1}{g/Ra+1}\left[T_0^{g/Ra+1} - T^{g/Ra+1}\right],$$

(585) $$\int_0^z T\,\rho\,d\,z = \frac{1}{g+Ra}\left(P_0\,T_0 - P_0\left(\frac{T}{T_0}\right)^{g/Ra} T\right) = \frac{1}{g+Ra}(P_0\,T_0 - P\,T).$$ Hence,

(586) $$\int_0^z T\,d\,m = \int_0^z T\,\rho\,d\,z = \frac{1}{g+\frac{k-1}{n\,k}}(P_0\,T_0 - P\,T).$$

The difference of the products of the pressure and the temperature at two points, multiplied by the coefficient depending on the n-coefficient of the gradient of the temperature between them, is the integral of this term. It is, however, much simpler to integrate by means of T and ρ for the stratum $(z_1 - z_0)$,

$$(587) \quad \int_0^s T\,d\,m = \int_0^s T\,\rho\,d\,z = T_{10}\,\rho_{10}\,(z_1 - z_0),$$

which gives close approximate values.

Case II. *Effect of an Adiabatic Expansion or Contraction in a Non-Adiabatic Temperature Gradient*

Since a mass in moving from one level to another level in the atmosphere begins to change adiabatically, while the prevailing temperature gradient is non-adiabatic, it becomes desirable to define the relation of these facts to the velocities which are immediately set up in the mixing medium. The equation (573) is to be evaluated under adiabatic conditions, by which it becomes,

$$(588) \quad \tfrac{1}{2}\,m_1\,q^2 = C p\left[T_1 - T_1\left(\frac{P_h}{P_1}\right)^{\frac{k-1}{k}}\right] m_1 - g\,h\,m_1.$$

From $\dfrac{P_h}{P_1} = \left(\dfrac{T_2}{T_1}\right)^{\frac{nk}{k-1}}$, we have, for $T_2 = T_1 - a\,h$,

$$(589) \quad \left(\frac{P_h}{P_1}\right)^{\frac{k-1}{k}} = \left(\frac{T_2}{T_1}\right)^{\frac{nk}{k-1}\cdot\frac{k-1}{k}} = \left(\frac{T_2}{T_1}\right)^n = \left(\frac{T_1 - a\,h}{T_1}\right)^n = \left(1 - \frac{a\,h}{T_1}\right)^n$$

From the binomial theorem, we have the formula,

$$(590) \quad (1 - x)^n = 1 - n\,x + \frac{n(n-1)}{2}\,x^2 = 1 - n\,x + \frac{n^2x^2}{2} - \frac{n\,x^2}{2},\ \text{so that}$$

$$(591) \quad \tfrac{1}{2}\,m_1\,q^2 = C p\left[T_1 - T_1\left(1 - n_1\frac{a\,h}{T_1} + n^2 . \tfrac{1}{2}\frac{a^2\,h^2}{T_1^2} - n . \tfrac{1}{2}\frac{a^2\,h^2}{T_1^2}\right]\right. m_1 - g\,h\,m_1.$$

(592) $\frac{1}{2} m_1 q^2 = g h m_1 - \frac{1}{2} \frac{g^2 h^2 m_1}{C p T_1} + \frac{1}{2} \frac{g a h^2 m_1}{T_1} - g h m_1.$

(593) $\frac{1}{2} m_1 q^2 = \frac{1}{2} \frac{g h^2 m_1}{T_1} \left(a - \frac{g}{Cp}\right) = \frac{1}{2} \frac{g h^2 m_1}{T_1} (a - a_0) =$

$$\frac{1}{2} \frac{g h^2 m_1}{T_1} \left(\frac{a_0}{n} - a_0\right).$$

The mass m_1 is driven from its position with a velocity energy inversely proportional to the temperature, so that warm air has less driving power than cold air. The drive depends upon the departure-ratio n and vanishes when $n = 1$, that is, for adiabatic expansion in an adiabatic gradient. When $a > a_0$ the mass m_1 is in unstable equilibrium, that is, too cold for its position and tends to fall. Example, for $n = 0.5$, $a = 19.747$, $a_0 = 9.87$. When $a < a_0$ the mass m_1 is in stable equilibrium. Example, for $n = 2$, $a = 4.94 < a_0 = 9.87$. It is not possible to drive the small mass m_1 through any great height in the atmosphere, because the differential energy of the expanding mass sets up minor whirls which tend to interchange the heat energy by mechanical effects and internal friction and radiation. The result is to change the gradient from a_0 to $\frac{a_0}{n}$. If the displacement of the mass m_1 takes place in the medium of gradient a, then the drive may be expressed by terms of the form

(594) $\frac{1}{2} m_1 q^2 = \frac{1}{2} g \frac{h^2}{T_1} m_1 \left(\frac{a_0}{n_1} - \frac{a_0}{n}\right) = \frac{1}{2} g \frac{h^2}{T} m_1 a_0 \left(\frac{n - n_1}{n n_1}\right),$

where n_1 is the effective temperature ratio of the moving mass, and n is that of the prevailing general temperature gradient.

There are two primary type cases of the distribution of the masses of different temperatures: (1) That in which they are superposed, and (2) that in which the masses are located side by side on the same levels.

CASE III. *The Overturn of Deep Strata in the Column*

Let the pressures, temperatures, and heights be arranged in the initial and final states as indicated in Fig. 20.

The Overturn of the Deep Strata in the Column

The greatest entropy in the initial state in 1 is less than the least in 2, so that the cold mass 1 will fall beneath the warm

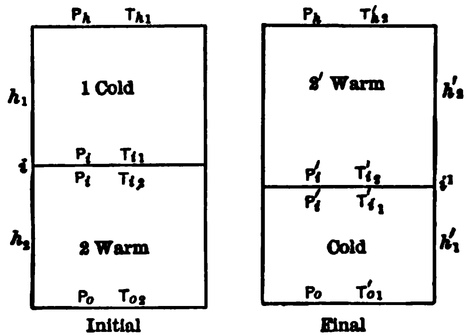


FIG. 20. The overturn of the deep strata in the column.

mass 2. The heights of the masses will change, as well as the pressures and temperatures. Assume that P_0, T_{02}, h_2, Ti_1, h_1 are known in the initial states.

We shall have the following formulas for computing the other required terms, in a non-adiabatic atmosphere.

$$(595)\quad P_i = P_0\left(\frac{T_{i2}}{T_{02}}\right)^{\frac{nk}{k-1}}. \qquad T_{i2} = T_{02} - \frac{g\,h_2}{n\,Cp}.$$

$$(596)\quad P_h = P_i\left(\frac{T_{h1}}{T_{i1}}\right)^{\frac{nk}{k-1}}. \qquad T_{h1} = T_{i1} - \frac{g\,h_1}{n\,Cp}.$$

Substitute in $Cp\left(\int T\,dm - \int T_1\,dm_1\right)$ successively, using (715).

$$(597)\quad \text{Initial.}\quad (V+U)_a = \tfrac{1}{2}\,m\,q^2 = Cp\int T\,dm =$$

$$\frac{Cp}{g+\frac{k-1}{nk}}\left(P_0\,T_{02} - P_i\,T_{i2} + P_i\,T_{i1} - P_h\,T_{h1}\right) + \text{const.}$$

$$(598)\quad \text{Final.}\quad (V+U)_e = \ldots\ldots\ldots =$$

$$\frac{Cp}{g+\frac{k-1}{nk}}\left(P_0\,T_{01}' - P_i'\,T_{i1}' + P_i'\,T_{i2}' - P_h\,T_{h2}'\right) + \text{const.}$$

(599) Kinetic energy $= (V + U)_a - (V + U)_e = \frac{1}{2} M q^2 =$

$$\frac{1}{2} \frac{P_0 - P_h}{g} q^2 \quad \text{(approx.)}.$$

(600) Heights. $h_1' = \frac{n\,Cp}{g} (T_{01}' - T_{i1}')$.

(601) $h_2' = \frac{n\,Cp}{g} (T_{i2}' - T_{h2}')$.

The approximate solution of this case (Margules) is

(602) Velocity. $\frac{1}{2} q^2 = \frac{g}{n} \frac{h_1 h_2 (T_{i2} - T_{i1})}{h_1 T_{i2} + h_2 T_{i1}}$.

Case IV. *The Transformation of Two Masses of Different Temperatures on the Same Levels into a State of Equilibrium.*

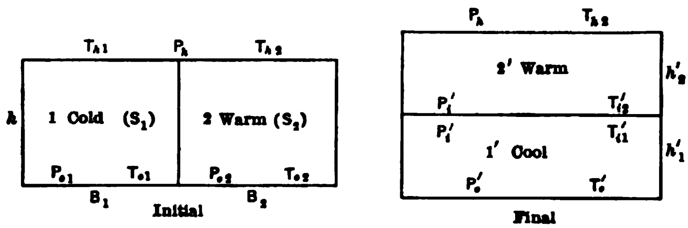


Fig. 21. Transformation of two masses on the same levels.

Transformation of Two Masses on the Same Levels

Given the initial data at the height h, T_{h1}, T_{h2}, P_h, the areas B_1, B_2, and the entropies $S_1 < S_2$. Hence by the formulas we shall have,

(603) $P_{01} = P_h \left(\frac{T_{01}}{T_{h1}}\right)^{\frac{nk}{k-1}} = P_h \left(1 + \frac{g\,h}{n\,Cp\,T_{h1}}\right)^{\frac{nk}{k-1}}$.

(604) $P_{02} = P_h \left(\frac{T_{02}}{T_{h2}}\right)^{\frac{nk}{k-1}} = P_h \left(1 + \frac{g\,h}{n\,Cp\,T_{h2}}\right)^{\frac{nk}{k-1}}$.

(605) $P_i' = P_h + \frac{1}{2} (P_{02} - P_h)$.

(606) $P_0' = P_h + \frac{1}{2} (P_{02} - P_h) + \frac{1}{2} (P_{01} - P_h)$.

(607) $T_{01} = T_{h1} \left(1 + \frac{g\,h}{n\,Cp\,T_{h1}}\right)$.

(608) $T_{02} = T_{h2} \left(1 + \frac{g\,h}{n\,Cp\,T_{h2}}\right)$.

(609) Initial. $(V + U)_a = \frac{Cp}{g} \frac{1}{1 + \frac{k-1}{nk}} \frac{B}{2} (P_{01} T_{01} - P_h T_{h1} +$

$P_{02} T_{02} - P_h T_{h2}) + \text{const.}$

(610) Final. $(V + U)_e = \frac{Cp}{g} \frac{1}{1 + \frac{k-1}{nk}} B (P_0' T_0' - P_i' T_i' +$

$P_i' T_{i2}' - P_h T_{h2}) + \text{const.}$

(611) Kinetic energy $= \frac{1}{2} M q^2 = (V + U)_a - (V + U)_e$.

(612) Mass. $M = \frac{B}{g} (P_0' - P_h)$.

(613) Height. $h_1' = \frac{Cp}{g} (T_0' - T_{i1}')$.

(614) $h_2' = \frac{Cp}{g} (T_{i2}' - T_{h2})$.

An approximate solution is given (Margules).

(615) Take $\tau = \frac{T_2 - T_1}{T}$, $T^2 = T_1 T_2$, $M = B P_h \frac{h}{RT} =$

$B \rho h$ (approx.).

(616) $\frac{1}{2} M q^2 = \frac{1}{2} M \frac{B_1 B_2}{B^2} g h \tau$.

These solutions must be handled cautiously in practice, because the internal motions of the atmosphere introduce elements of pressure, temperature, and velocity which it is very difficult to follow, and take account of in forming the elements of the integrals, and there is no term for the radiation.

CASE V. *For local changes between two strata of different temperatures, where on the boundary the pressure* $P = P_1' = P_2'$ *and the temperature is discontinuous*

Take the following conditions:

(617) $m_2, P_2 T_2, P_2' = P_2 + g m_1, T_2' = T_2 \left(\frac{P_2 + g m_1}{P_2}\right)^{\frac{k-1}{nk}}$.

(618) $m_1, P_1 T_1, P_1' = P_1 - g m_2, T_1' = T_1 \left(\frac{P_1 - g m_2}{P_1}\right)^{\frac{k-1}{nk}}$.

The condition of equilibrium becomes, for $P_1' = P_2' = P$,

(619) $$\text{Kinetic energy} = Cp\,[m_1(T_1 - T_2') + m_2(T_2 - T_2')]$$

(620) $$= Cp\left[m_1\left(T_1 - T_1 + T_1\frac{R}{n\,Cp}\frac{g\,m_2}{P}\right) + m_2\left(T_2 - T_2 - T_2\frac{R}{n\,Cp}\frac{g\,m_1}{P}\right)\right].$$

(621) $$= m_1 m_2 \frac{R\,g}{n\,P}(T_1 - T_2) = m_1 m_2 \frac{g}{n}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right).$$

(622) Since $\dfrac{R\,T_1}{P_1} = \dfrac{1}{\rho_1}$ and $\dfrac{R\,T_2}{P_2} = \dfrac{1}{\rho_2}$, therefore,

(623) $$\tfrac{1}{2} M\,q^2 = m_1 m_2 \frac{g}{n}\frac{\rho_2 - \rho_1}{\rho_1\,\rho_2}.$$

The kinetic energy inducing an interchange is proportional to the difference of the densities, and inversely proportional to the product of the densities. Hence, if strata of different densities are flowing over one another in the general circulation, which is temporarily stratified, these two strata tend to mix by interpenetration according to this law.

There are numerous other cases which can be worked out in accordance with the law which may be assumed for the distribution of the temperature in a vertical and in a horizontal direction.

Compare "Ueber die Energie der Stürme," von Max Margules. Jahrbuch der k.k. Cent.-Anst. für Meteo. u. Erdeng. Wien, 1903.

"The Thermodynamics of the Atmosphere," F. H. Bigelow. *Monthly Weather Review*, 1906, and Bulletin W. B. No. 372.

The General Circulation on a Hemisphere of the Earth's Atmosphere

While it is impracticable to take up the problems of the general circulation with the purpose of forming integrals that will take account of the entire circulation, there are yet a few propositions which are of interest in the premises.

Resume equation $(414)_2$, and limit it by assuming symmetry about the axis of rotation, so that $-\dfrac{1}{\rho}\dfrac{\partial P}{\partial y} = 0$; also, by omitting the small term in w, so that we shall have as a special case, wherein the vertical current and the friction are omitted from the general motion,

$$(624)\quad \cos\theta\,(2\,\omega_3 + \nu)\,u + \frac{d\,v}{d\,t} = 2\cos\theta\,(\omega_3+\nu)\,u - \nu\cos\theta\,.\,u + \frac{d\,v}{d\,t} = 0.$$

Multiply this equation by $r\sin\theta$, and substitute from (413),

$$(625)\quad \frac{d\,v}{d\,t} - \nu\cos\theta\,.\,u = r\sin\theta\frac{\partial\nu}{\partial t}, \text{ so that,}$$

$$(626)\quad 2\,r\sin\theta\,(\omega_3 + \nu)\,r\cos\theta\frac{\partial\theta}{\partial t} + (r\sin\theta)^2\frac{\partial\nu}{\partial t} = 0.$$

Integrating for *each* gyrating particle,

$$(627)\quad r^2\sin^2\theta\,(\omega_3 + \nu) = c.$$

Let C = constant for the *entire* rotating mass, if ν_0 is the initial relative angular velocity,

$$(628)\quad C = \frac{\int c\,d\,m}{m} = \frac{\int r^2\sin^2\theta\,(\omega_3 + \nu_0)\,d\,m}{m} = \tfrac{2}{3}\,r^2\,(\omega_3\,\nu_0').$$

This is equal to the moment of inertia of the entire mass at the distance $\frac{2}{3}\,r$. If the initial state is that of rest on the rotating earth $\nu_0' = 0$. Finally, we shall have:

$$(629)\quad r^2\sin^2\theta\left(\omega_3 + \frac{v}{r\sin\theta}\right) = \tfrac{2}{3}\,r^2, \text{ and}$$

$$(630)\quad r\,\omega_3\sin\theta + v = \tfrac{2}{3}\frac{r\,\omega_3}{\sin\theta}, \text{ or}$$

$$(631)\quad v = \left(\tfrac{2}{3}\frac{1}{\sin\theta} - \sin\theta\right)r\,\omega_3.$$

This is the eastward velocity at the north polar distance θ.

(632) If $v = 0$, $\sin^2\theta = \frac{2}{3}$, $\sin\theta = 0.8165$, $\theta = 54°\,44'$.

At latitude $\phi = 90 - \theta = 90 - 54°\,44' = 35°\,16'$ the eastward velocity vanishes at the surface of the earth. Observations indicate that $v = 0$ extends upward in a direction sloping towards the equator from latitude 35° north and south, its position being indefinite above 20,000 meters.

The equation (624) can be obtained as follows: Assume the vortex principle of the conservation of the momenta of inertia,

$$(633) \quad \varpi^2 (\omega_3 + \nu) = (r \sin \theta)^2 (\omega_3 + \nu) = c \text{ a constant.}$$

This is not strictly true in the atmosphere, because it is not circulating in a perfect vortex, and this faulty assumption has been generally made in discussing this subject. How far it departs from a vortex law remains to be determined by the observations. Differentiate, divide by $r \sin \theta \, d\, t$, and we obtain

$$(634) \quad 2 \cos \theta \, (\omega_3 + \nu) \, r \frac{d\,\theta}{d\,t} + r \sin \theta \frac{d\,v}{d\,t} = 0.$$

Since $r \dfrac{d\,\theta}{d\,t} = u$, we find from (624),

$$(635) \quad \nu \cos \theta . \; u - \frac{d\,v}{d\,t} + r \sin \theta \frac{d\,v}{d\,t} = 0,$$

and this is the same as (625).

Ferrel discusses these equations, and gives some approximately correct views regarding the general circulation. Oberbeck's treatment embraces the three equations of motion, and the solution approaches more closely to the flow of currents actually observed. The complete integration of the system is, however, more complex than has been admitted, and the problem awaits a better treatment. The actual velocities and direction of motion, together with the temperatures, must be so handled as to embrace the general and the local circulations in a single comprehensive solution.

Three Cases of the Slope of the Temperature Gradients and the Resulting Velocity of the East and West Circulations

In the earth's atmosphere there are three general cases of the distribution of the temperature gradients and the resulting circulation which can be distinguished, though the solution will not be complete until the radiation term has been accounted for in the equations of motion. These cases are: (1) for the eastward drift in the temperate zone where the velocity increases upwards, while the temperature decreases towards the pole in a line parallel to the axis of the earth's rotation; (2) for the lower levels of the westward drift in the tropics, up to an altitude of about 5,000 meters, wherein the westward motion increases with the height, while the temperature increases towards the pole; (3) for the upper layers of the westward drift, above 5,000 meters into the isothermal region, wherein the velocity decreases with the height, while the temperature increases towards the pole. The case (3) seems to agree with the conditions observed in the atmosphere of the sun, which has decreasing velocity from the equator to the pole, and decreasing velocity from the surface upwards in all latitudes, accompanied by increasing temperatures towards the pole. The observations in the tropics on the cloud velocities give an increasing velocity westward to 5,000

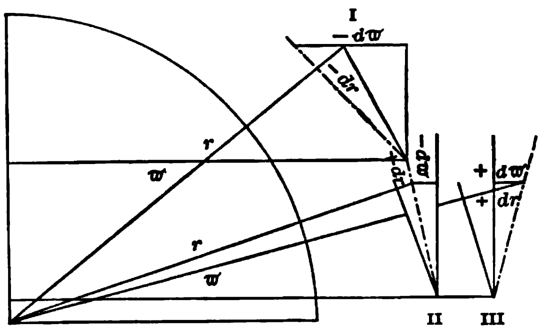


FIG. 22. The relative values of dr and $d\varpi$ in three cases.

meters, then decreasing to 11,000, then increasing to the limit of balloon observations, so that cases (2) and (3) alternate to some extent.

ϖ = the distance from the axis of rotation of the earth.

r = the radius from the center of the earth.

Draw a tangent to the circle at the initial point of the isotherm. Draw $d\varpi$ and dr to second points on the same isotherm, to show its slope relative to the horizon and axis of rotation. We have to determine the relation of the temperatures and the velocities of motion in space to each other at any point in the earth's atmosphere.

Take the general integral of motion (417), omitting the

$(Q + J)$ term, and supply the centrifugal force in the gravitation term, $-\frac{1}{2} v_0^2$, where v_0 is the linear velocity of the rotating surface at the given latitude, and we obtain,

$$(636) \quad -\int \frac{dP}{\rho} = \tfrac{1}{2}(u^2 + v^2 + w^2) - \tfrac{1}{2} v_0^2 + g\,r + \text{constant}.$$

Substitute $\frac{1}{\rho} = \frac{R\,T}{P}$, and pass to logarithms, also put

$g\,r = \frac{g_0 R^2}{r}$ from the general law of gravity.

$$(637) \quad -\log P\,.\,R\,T = \tfrac{1}{2}(u^2+w^2) + \tfrac{1}{2}(v^2-v_0^2) + \frac{g_0 r_0^2}{r} + C.$$

We give different values of the temperature (T_1, T_2) to two adjacent strata flowing over each other at different velocities (v_1, v_2), but since the pressures cannot be discontinuous at the bounding surface, we take $P_1 = P_2$. Hence, by substitution in two strata, and transformation of the terms for differentiation,

$$(638) \quad \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\frac{g_0 r_0^2}{r} = \frac{\frac{1}{2}(v_2^2 - v_0^2)}{T_2} - \frac{\frac{1}{2}(v_1^2-v_0^2)}{T_1} + \frac{\frac{1}{2}(u_2^2+w_2^2)}{T_2} - \frac{\frac{1}{2}(u_1^2 - w_1^2)}{T_1} + \frac{C_2}{T_2} - \frac{C_1}{T_1}.$$

This is the general equation to be fulfilled at every point. Now differentiate (637) to r, the change along the radius, for two adjacent strata at pressures P_1, P_2, and we have:

$$(639) \quad -R\frac{d(\log P_1 - \log P_2)}{d\,r} = -g\left(\frac{1}{T_1} - \frac{1}{T_2}\right) = -g\,\frac{(T_2 - T_1)}{T_1 T_2},$$

omitting the small terms in u and w.

Again differentiate (637) to ϖ, holding the angular momentum $(v\,\varpi)$ constant in each stratum. At the surface of the earth the velocity $v_0^2 = \omega_3^2\,\varpi^2$. Hence,

$$(640) \quad \frac{d\,v_0^2}{d\,\varpi} = 2\,\omega_3^2\,\varpi = \frac{2\,\omega_3^2\,\varpi^2}{\varpi} = \frac{2\,v_0^2}{\varpi}.$$ Using this form,

and differentiating for two adjacent strata,

$$(641) \quad -R\frac{d(\log P_1 - \log P_2)}{d\,\varpi} = \frac{1}{\varpi}\left(\frac{v_1^2 - v_0^2}{T_1} - \frac{v_2^2 - v_0^2}{T_2}\right) = \frac{1}{\varpi}\left[\frac{(v_1^2 - v_0^2)\,T_2 - (v_2^2 - v_0^2)\,T_1}{T_1 T_2}\right].$$

Divide (641) by (639) and the ratio $\frac{dr}{d\varpi}$ becomes,

$$(642)\quad \frac{dr}{d\varpi} = -\frac{1}{g\varpi}\left[\frac{(v_1^2-v_0^2)\,T_2-(v_2^2-v_0^2)\,T_1}{T_2-T_1}\right].$$

This equation connects the velocities and temperatures with the slope of the isotherms, and it is capable of three solutions which are expressed as follows:

Case I	Case II	Case III
$\frac{-dr}{-d\varpi} = -\frac{(+)}{(-)}$	$\frac{+dr}{-d\varpi} = -\frac{(+)}{(+)}$	$\frac{+dr}{+d\varpi} = -\frac{(-)}{(+)}$
$(v_1^2-v_0^2)\,T_2 > (v_2^2-v_0^2)\,T_1$	$(v_1^2-v_0^2)\,T_2 > (v_1^2-v_0^2)\,T_1$	$(v_1^2-v_0^2)\,T_2 < (v_2^2-v_0^2)\,T_1$
$T_2 < T_1$	$T_2 > T_1$	$T_2 > T_1$
East	West (lower)	West (upper) and Sun

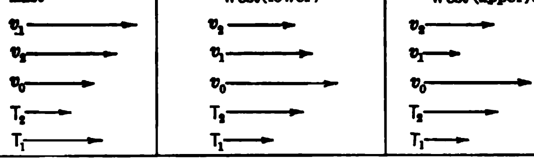


FIG. 23. The relative values of v_0, v_1, v_2, T_1, T_2 in three Cases.

If $dr = 0$, and the isotherm is parallel to the surface, it follows that $(v_1^2 - v_0^2)\,T_2 = (v_2^2 - v_0^2)\,T_1$ so that the crossed products of the square of the relative velocities at any point in the atmosphere by the alternating temperatures of the two adjacent strata are equal. The warm stratum assumes greater velocity than the cold stratum, in order to maintain a gradual change in the value of the vertical hydrostatic pressure, such as was developed in Chapter II. If dr changes from 0, in the three cases described, and typically illustrated, the temperature gradients take on slopes that respectively balance the velocities of the air movements, generally above the series of tangents to the horizon in the tropics, but below them in middle latitudes, as have been found from the direct observations in balloon ascensions. In Chapter II, it has been shown how powerfully the $(Q_1 - Q_0)$, the change of the heat contents per unit mass from one level to another reacts upon the velocity system, so that this problem cannot be fully solved through velocity and temperature functions. These theorems can be extended to very useful inferences in the case of the sun where velocities can be measured, but where it is very difficult to determine the absolute temperatures prevailing in different strata.

CHAPTER IV

Examples of the Construction of Vortices in the Earth's Atmosphere

An extensive computation on vortices has been published in the *Monthly Weather Review*, October, 1907, and subsequent numbers, giving in sufficient detail the method of handling the data. These comprise the funnel-shaped vortex of the Cottage City waterspout, August 19, 1896, the dumb-bell-shaped vortex of the same Cottage City waterspout, the truncated dumb-bell-shaped vortex of the St. Louis tornado, May 27, 1896, the De Witte typhoon, August 1–3, 1901, the impure dumb-bell vortex in the ocean cyclone, October 11, 1905, and the very imperfect vortices of the land cyclones of the United States. In Figs. 24, 25, 26, are given typical ($\varpi . z$) lines in the Cottage City waterspout, the St. Louis tornado, and the De Witte typhoon, respectively, to which further references will be made.

In the funnel-shaped vortex tube of the Cottage City waterspout, the plane of reference is at the base of the cloud, 1,100 meters above the sea level, and the axis extends downwards, this being the convenient form for tornadoes generally. In the dumb-bell-shaped vortex there are two planes of reference, and the lower one is placed below the sea level while the upper one is placed at the cloud level. The axis between these planes is divided into 180 degrees so that, $a = \frac{180}{1,200} = 0.15$, this being the value in the current function ψ for this case. It has been found that these vortices are generated at the cloud base by the thermodynamic action of strata of different temperatures, and that they are propagated downwards to the sea level or to the surface of the ground. These vortices seem to be cut off or truncated at some distance above the lower plane of reference, and on this supposition the vortex laws, when applied to the observed phenomena, appear to meet satisfactorily all the requirements of the problem. A few details of the computations will be introduced in this connection.

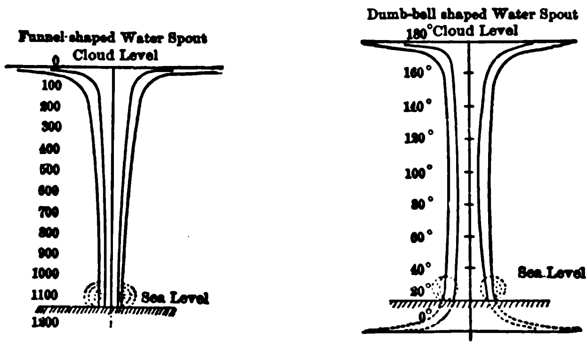
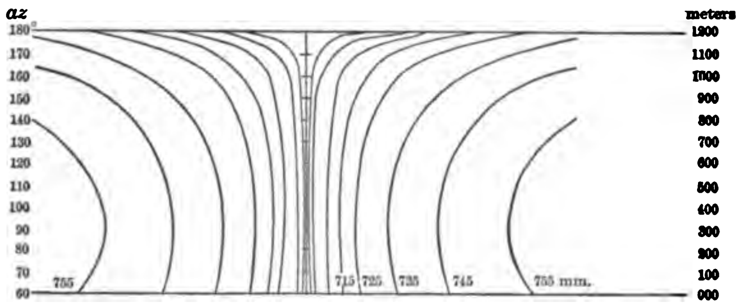


FIG. 24. The Cottage City, Mass., waterspout, August 19, 1896.



St. Louis Tornado, May 27, 1896.
FIG. 25. Illustrating the truncated dumb-bell-shaped vortex.

The vertical ordinate is magnified ten times.

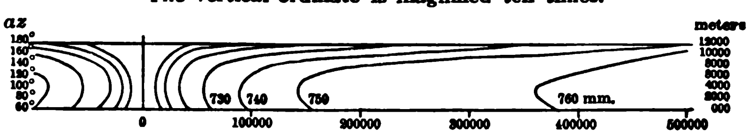


FIG. 26. The De Witte typhoon, August 1–3, 1901.

In constructing a vortex of either type it is necessary to know two facts from observations: (1) the tangential velocity v at a point whose radius is ϖ in meters from the axis, on a plane defined by z in the funnel-shaped vortex, or by az in the dumb-bell-shaped vortex; (2) the ratio ρ, which is the ratio of the successive radii ϖ in the tubes. By a series of studies on the spacing of the isobars in the De Witte typhoon, the St. Louis tornado, and the ocean cyclone of October 11, 1905, it was found that $\log \rho = 0.20546$ seems to comply with the positions of the tubes in these vortices as developed in the earth's atmosphere. In the cyclones and anticyclones there is a wide departure from this simple constant ratio, which indicates that another source of energy is at work besides the one generating these simple vortices, but this will require a fuller explanation. In Tables 45–53, and on Figs. 24, 25, 26, are given the results of the computations in sufficient detail to illustrate the scope of the formulas, the dimensions of the vortices, and the velocities, together with the angles of the helices which they make in the tubes. The funnel-shaped tube of Fig. 24 is constructed from $\varpi\ z$ in Table 45, using the tube (1); the truncated dumb-bell-shaped vortex of the St. Louis tornado is constructed from Table 48. An examination of the tables of the velocities and the angles suggests numerous remarks on their relations, but as they can be very clearly perceived it is not necessary to write them down. The vortices differ from one another in their dimensions, the waterspout, tornado, and hurricane being illustrations of the dumb-bell vortex. The cyclone shows a close relationship to this type of vortex, but it is distinctly modified by a different distribution of the thermal energy. The meteorological data are so extensive as to make it impracticable to reproduce them in this Treatise.

Table 45 contains the necessary initial data and the formulas for developing a funnel-shaped vortex in all its tubes from the outer to the inner in succession. Taking the assumed data $z = 100$, $\varpi = 60$, $v = 6.67$ m/sec, $\psi = v\ \varpi$, we proceed through the formula to construct for tube (1), which has the largest radius, the constant of that tube C, the velocities u, v, w, and the horizontal angle i and the vertical angle η as defined in Fig. 14 for cylindrical co-ordinates, and Fig. 18 for vortices. Then follows the successive application of the formulas (535) to (550) by which the data of tubes (2), (3), (4), (5), (6) are computed.

Thus by subtracting 0.20546 from $\log \varpi$, in succession, the radii of the other tubes $\varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6$, are obtained; similarly, the $\log \rho$ as indicated is to be applied for C, u, v, w, i, η.

Table 48 contains the initial assumed data for the dumb-bell-shaped vortex of the St. Louis tornado, truncated on the plane $az = 60°$. Thus, for $az = 60°$, $i = -30°$, $\varpi = 960$, $v = 13.1$ m/sec, we find in succession, $a = 0.100$, $A, u, v, w, \varpi, i, \eta$, on the tube (1), and on tubes (2), (3), (4), (5), (6), by applying $\log \rho$ or its multiples as indicated by the working group of formulas (535)–(550). In the same manner we proceed with the De Witte typhoon, the ocean cyclone, and similar highly developed vortices. The land cyclone and anticyclone are imperfect vortices, and they involve a system of hydrodynamic stream lines which are highly complex in their origin and development. It must be constantly remembered that the important radiation terms do not appear in these vortex formulas and examples, so that a fuller treatment would be much more complex than the one here briefly summarized. A further illustration will be added in discussing the origin of the cyclone.

TABLE 45

THE COTTAGE CITY WATERSPOUT, AUGUST 19, 1896

The Funnel-shaped Vortex Tubes, $\psi = C \varpi^2 z$

Collection of the Constants and Working Formulas

Assumed data. $z = 100$ meters, distance below cloud plane
$\varpi = 60$ meters, radius of cloud sheath.
$v = 6.67$ m/sec, tangential velocity at (z, ϖ).

Formulas. $C = \frac{\psi}{\varpi^2 z}$ the constant for each tube.

$u = C\varpi$ the radial velocity.

$v = \frac{\psi}{\varpi}$ the tangential velocity.

$w = -2Cz$ the vertical velocity.

$\psi = v\varpi$ constant. $\log \psi = 2.60206$.

The ratio of the successive radii, $\rho = \frac{\varpi_n}{\varpi_{n+1}}$. $\log \rho = 0.20546$.

The successive radii, $\log \varpi_{n+1} = \log \varpi_n - \log \rho$.

The successive velocities, $\log u_{n+1} = \log u_n + \log \rho$.

$\log v_{n+1} = \log v_n + \log \rho$.

$\log w_{n+1} = \log w_n + 2 \log \rho$.

The successive angles, $\log \tan i_n = \text{constant}$. $\tan i = \dfrac{u}{v}$

$$\log \tan \eta_{n+1} = \log \tan \eta_n + \log \rho. \quad \tan \eta = \frac{-w}{v \sec i}.$$

TABLE 46

The Values of ϖ, C, u, v, w, on the Plane $z = 100$

Formula	(1)	(2)	(3)	(4)	(5)	(6)	Compare formulas (535)–(550)	log ρ
log ϖ	1.77815	1.57269	1.36723	1.16177	0.95631	0.75085	−	0.20546
ϖ	60.0	37.4	23.3	14.5	9.0	5.6		
log C	7.04576	7.45668	7.86760	8.27852	8.68944	9.10036	+	0.41092
C	0.001111	0.002862	0.007372	0.01899	0.04891	0.12600		
log u	8.82391	9.02937	9.23483	9.44029	9.64575	9.85121	+	0.20546
u	0.06667	0.1070	0.1717	0.2756	0.4423	0.7099		
log v	0.82391	1.02937	1.23483	1.44029	1.64575	1.85121	+	0.20546
v	6.67	10.70	17.17	27.56	44.23	70.99		
log w	−9.34679	−9.75771	−0.16863	−0.57955	−0.99047	− 1.40139	−	0.41092
w	−0.222	−0.572	−1.474	−3.798	−9.783	−25.199		
log tan i	8.00000	8.00000	8.00000	8.00000	8.00000	8.00000		
i	0° 34′	0° 34′	0° 34′	0° 34′	0° 34′	0° 34′		
log tan η	8.52288	8.72834	8.93880	9.13926	9.34472	9.55018	+	0.20546
η	1 55	3 4	4 54	7 51	12 28	19 33		

TABLE 47

The Values of ϖ, u, v, w, i, η on Several Planes

The Radii ϖ

z	(1)	(2)	(3)	(4)	(5)	(6)
0	∞	∞	∞	∞	∞	∞
1	600.0	373.8	232.9	145.1	90.4	56.3
10	189.7	118.2	73.7	45.9	28.6	17.8
25	120.0	74.8	46.6	29.0	18.1	11.3
50	84.9	52.9	32.9	20.5	12.8	8.0
100	60.0	37.4	23.3	14.5	9.0	5.6
200	42.4	26.4	16.5	10.3	6.4	4.0
300	34.6	21.6	13.5	8.4	5.2	3.3
400	30.0	18.7	11.7	7.3	4.5	2.8
500	26.8	16.7	10.4	6.5	4.0	2.5
700	22.7	14.1	8.8	5.5	3.4	2.1
900	20.0	12.5	7.8	4.8	3.0	1.9
1100	18.1	11.3	7.0	4.4	2.7	1.7

TABLE 47.—CONTINUED

z	(1)	(2)	(3)	(4)	(5)	(6)
			The Radial Velocity u			
0	∞	∞	∞	∞	∞	∞
1	0.667	1.070	1.717	2.756	4.423	7.099
10	0.211	0.339	0.544	0.874	1.402	2.250
25	0.133	0.214	0.343	0.551	0.885	1.420
50	0.094	0.151	0.243	0.390	0.626	1.004
100	0.067	0.107	0.172	0.276	0.442	0.710
200	0.047	0.076	0.121	0.195	0.313	0.502
300	0.039	0.062	0.099	0.159	0.255	0.410
400	0.033	0.054	0.086	0.138	0.221	0.355
500	0.030	0.048	0.077	0.123	0.198	0.318
700	0.025	0.040	0.065	0.104	0.167	0.268
900	0.022	0.036	0.057	0.092	0.147	0.237
1100	0.020	0.032	0.052	0.083	0.133	0.214
			The Tangential Velocity v			
0	0	0	0	0	0	0
1	0.7	1.1	1.7	2.8	4.4	7.1
10	2.1	3.4	5.4	8.7	14.0	22.5
25	3.3	5.4	8.6	13.8	22.1	35.5
50	4.7	7.6	12.1	19.5	31.3	50.2
100	6.7	10.7	17.2	27.6	44.2	71.0
200	9.4	15.1	24.3	39.0	62.6	100.4
300	11.6	18.5	29.7	47.7	76.6	122.9
400	13.3	21.4	34.3	55.1	88.5	142.0
500	14.9	23.9	38.4	61.6	98.9	158.7
700	17.6	28.3	45.4	72.9	117.0	187.8
900	20.0	32.1	51.5	82.7	132.7	213.0
1100	22.1	35.5	57.0	91.4	146.7	235.7
			The Vertical Velocity w			
0	0	0	0	0	0	0
1	−0.0022	−0.0057	− 0.0147	− 0.0380	− 0.0978	− 0.2520
10	−0.0222	−0.0572	− 0.1474	− 0.3798	− 0.9783	− 2.520
25	−0.0556	−0.1431	− 0.3686	− 0.9495	− 2.446	− 6.300
50	−0.1110	−0.2862	− 0.7372	− 1.889	− 4.891	− 12.60
100	−0.222	−0.572	− 1.474	− 3.798	− 9.785	− 25.20
200	−0.444	−1.145	− 2.949	− 7.596	− 19.57	− 50.40
300	−0.667	−1.717	− 4.423	−11.39	− 29.35	− 75.60
400	−0.889	−2.290	− 5.898	−15.19	− 39.13	−100.80
500	−1.111	−2.862	− 7.372	−18.99	− 48.91	−126.00
700	−1.556	−4.007	−10.32	−26.59	− 68.48	−176.40
900	−2.000	−5.152	−13.27	−34.18	− 88.05	−226.80
1100	−2.444	−6.297	−16.22	−41.78	−107.61	−277.20

The Horizontal Angle i

	°	′		°	′
0	90	0	.........Constant..................	90	0
10	5	43	"	5	43
50	1	9	"	1	9
100	0	34	"	0	34
300	0	11	"	0	11
500	0	7	"	0	7
700	0	5	"	0	5
900	0	4	"	0	4
1100	0	3	"	0	3

The Vertical Angle η

	°	′	°	′	°	′	°	′	°	′	°	′
0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	36	0	58	1	33	2	29	3	59	6	22
50	1	18	2	10	3	28	5	34	8	53	14	5
100	1	55	3	4	4	54	7	51	12	28	19	33
300	3	18	5	18	8	28	13	25	20	58	31	35
500	4	16	6	49	10	52	17	8	26	19	38	26
700	5	2	8	3	12	48	20	2	30	20	43	12
900	5	43	9	7	14	27	22	28	33	34	46	48
1100	6	19	10	4	15	54	24	34	36	16	49	39

All the data on the level z = 100 meters have been computed in Table 46.

The St. Louis Tornado, May 27, 1896

The Truncated Dumb-bell-shaped Vortex Tubes, $\psi = A^2 \varpi^2 \sin az$

TABLE 48

COLLECTION OF THE CONSTANTS AND WORKING FORMULAS

Assumed data $az = 60°$ $i = -30°$ on the sea-level plane.

$\varpi = 960$ meters, radius of the outer tube.

$v = 13.1$ m/sec. tangential velocity at $(\varpi \cdot az)$.

$a = \dfrac{180}{1200+600} = 0.100$ $\log \rho = 0.20546$.

$\log a \sin az = 8.93753$ $\log a \cos az = 8.69897$.

$A = \dfrac{v}{a \varpi \sin az}$, constant for each tube.

$u = -A\, a\, \varpi \cos az$, the radial velocity.

$v = \dfrac{a\,\psi}{\varpi}$, the tangential velocity.

$w = +A\, a\, \varpi \sin az$, the vertical velocity.

$\varpi = \left(\dfrac{a\,\psi}{A\, a \sin az}\right)^{\frac{1}{2}}$, the radius on different levels.

$$\tan i = \tan\,(90 + az) = -\cot\,az = \frac{u}{v}.$$

$$\tan \eta = \frac{w}{v \sec i}, \text{ vertical angle.}$$

$$q = v \sec i \sec \eta, \text{ total velocity.}$$

TABLE 49

THE VALUES OF ϖ, A, u, v, w ON THE PLANE $az = 60°$

Formula	(1)	(2)	(3)	(4)	(5)	(6)		log ρ
log ϖ	2.98227	2.77681	2.57135	2.36589	2.16043	1.95497	Compare formulas (535)–(550)	− 0.20546
ϖ	960.0	598.2	372.7	232.2	144.7	90.2		
log A	9.19672	9.60764	0.01856	0.42948	0.84040	1.25182		+ 0.41092
A	0.1573	0.4052	1.0437	2.6883	6.9247	17.8871		
log u	−0.87796	−1.08342	−1.28888	−1.49434	−1.69980	−1.90526		− 0.20546
u	−7.6	−12.1	−19.5	−31.2	−50.1	−80.4		
log v	1.11652	1.32198	1.52744	1.73290	1.93836	2.14382		+ 0.20546
v	13.1	21.0	33.7	54.1	86.8	139.3		
log w	9.43528	9.84620	0.25712	0.66804	1.07896	1.48988		+ 0.41092
w	0.27	0.70	1.81	4.66	12.00	30.89		
log tan i	−9.76144	....	Constant	....	....	−9.76144		.....
i	−30°	....	....	....	....	−30°		.....
log tan η	8.25629	8.46175	8.66721	8.87267	9.07813	9.28359		+ 0.20546
η	1° 2′	1° 39′	2° 40′	4° 16′	6° 50′	10° 53′		

TABLE 50

THE VALUES OF ϖ, u, v, w, i, η ON SEVERAL PLANES

az	(1)	(2)	(3)	(4)	(5)	(6)
			The Radii ϖ			
180°	∞	∞	∞	∞	∞	∞
170	2143.9	1335.8	832.3	518.6	323.1	201.3
160	1527.6	951.8	593.0	369.5	230.2	143.5
150	1263.4	787.2	490.5	305.6	190.4	118.6
140	1114.3	694.3	432.6	269.5	167.9	104.6
130	1020.7	636.0	396.3	246.9	153.8	95.9
120	960.0	598.1	372.7	232.2	144.7	90.2
110	921.6	574.2	357.8	222.9	138.9	86.5
100	900.2	560.9	349.5	217.8	135.7	84.5
90	893.4	556.6	346.8	216.1	134.6	83.9
80	900.2	560.9	349.5	217.8	135.7	84.5
70	921.6	574.2	357.8	222.9	138.9	86.5
60	960.0	598.1	372.7	232.2	144.7	90.2

TABLE 50.—CONTINUED

z	(1)	(2)	(3)	(4)	(5)	(6)
			The Radial Velocity *u*			
180°	∞	∞	∞	∞	∞	∞
170	33.2	53.3	85.5	137.3	220.3	353.6
160	22.6	36.2	58.2	93.4	149.8	240.4
150	17.2	27.6	44.3	71.2	114.2	183.3
140	13.4	21.6	34.6	55.5	89.1	143.0
130	10.3	16.6	26.6	42.7	68.5	109.9
120	7.6	12.1	19.5	31.2	50.1	80.4
110	5.0	8.0	12.8	20.5	32.9	52.8
100	2.5	4.0	6.3	10.2	16.3	26.2
90	0.0	0.0	0.0	0.0	0.0	0.0
80	−2.5	− 4.0	− 6.3	−10.2	−16.3	−26.2
70	−5.0	− 8.0	−12.8	−20.5	−32.9	−52.8
60	−7.6	−12.1	−19.5	−31.2	−50.1	−80.4
			The Tangential Velocity *v*			
180°	0	0	0	0	0	0
170	5.9	9.4	15.1	24.2	38.9	62.4
160	8.2	13.2	21.2	34.0	54.5	87.5
150	9.9	16.0	25.6	41.1	65.9	105.8
140	11.3	18.1	29.0	46.6	74.8	120.0
130	12.3	19.7	31.7	50.9	81.6	131.0
120	13.1	21.0	33.7	54.1	86.8	139.3
110	13.6	21.9	35.1	56.3	90.4	145.1
100	13.9	22.4	35.9	57.7	92.5	148.5
90	14.1	22.6	36.2	58.1	93.2	149.6
80	13.9	22.4	35.9	57.7	92.5	148.5
70	13.6	21.9	35.1	56.3	90.4	145.1
60	13.1	21.0	33.7	54.1	86.8	139.3
			The Vertical Velocity *w*			
180°	0	0	0	0	0	0
170	0.06	0.14	0.36	0.93	2.41	6.20
160	0.11	0.28	0.71	1.84	4.74	12.20
150	0.16	0.41	1.04	2.69	6.93	17.84
140	0.20	0.52	1.34	3.46	8.90	22.93
130	0.24	0.62	1.60	4.22	10.86	27.96
120	0.27	0.70	1.81	4.66	11.99	30.90
110	0.30	0.76	1.96	5.05	13.01	33.52
100	0.31	0.80	2.06	5.30	13.64	35.13
90	0.32	0.81	2.09	5.38	13.85	35.67
80	0.31	0.80	2.06	5.30	13.64	35.13
70	0.30	0.76	1.96	5.05	13.01	33.52
60	0.27	0.70	1.81	4.66	11.99	30.90

The Horizontal Angle i

180°	+90°	.........Constant..................	+90°
160	+70	"	+70
140	+50	"	+50
120	+30	"	+30
100	+10	"	+10
90	0	"	0
80	−10	"	−10
70	−20	"	−20
60	−30	"	−30

The Vertical Angle η

180°	0°	0′	0°	0′	0°	0′	0°	0′	0°	0′	0°	0′
170	0	6	0	9	0	14	0	23	0	37	0	59
160	0	15	0	25	0	40	1	4	1	42	2	44
150	0	27	0	44	1	10	1	52	3	0	4	49
140	0	40	1	4	1	42	2	44	4	23	7	0
130	0	52	1	23	2	13	3	38	5	49	9	17
120	1	2	1	39	2	40	4	16	6	50	10	53
110	1	10	1	52	3	0	4	49	7	42	12	15
100	1	15	2	1	3	14	5	10	8	16	13	7
90	1	17	2	3	3	18	5	17	8	27	13	25
80	1	15	2	1	3	14	5	10	8	16	13	7
70	1	10	1	52	3	0	4	49	7	42	12	15
60	1	2	1	39	2	40	4	16	6	50	10	53

The data on the level $a\,z$ have been computed in Table 49.

TABLE 51

THE DE WITTE TYPHOON, AUGUST 1–3, 1901,

IN THE CHINA SEA

Results from the plane $a\,z = 60°$

Initial z = 12000 meters $a = \dfrac{180°}{12000 + 600} = 0.010$

z	(1)	(2)	(3)	(4)	(5)	(6)
A	.002016	.005193	.013375	.034452	.088744	.248922

The Radii ϖ in Meters

$a z$ = 180°	∞	∞	∞	∞	∞	∞
170	887600	534338	332938	207443	129253	80534
160	611071	380742	237232	147813	92098	57384
150	505389	314900	196205	122250	76172	47460
140	445740	277727	173044	107820	67180	41858
130	408310	254406	158515	98766	61539	38344
120	384018	239272	149083	92890	57878	36062
110	368655	229700	143120	89174	55563	34619
100	360117	224379	139803	87108	54275	33818
90	357367	222663	138735	86444	53860	33559
80	360117	224379	139803	87108	54275	33818
70	368655	229700	143120	89174	55563	34619
60	384018	239272	149083	92890	57878	36062

The Radial Velocity u in Meters per Second

$a z$ = 180°	∞	∞	∞	∞	∞	∞
170	17.03	27.32	43.86	70.38	112.96	181.30
160	11.58	18.58	29.82	47.85	76.80	123.27
150	8.82	14.16	22.72	36.47	58.53	93.94
140	6.88	11.05	17.73	28.46	45.67	73.30
130	5.29	8.49	13.63	21.87	35.10	56.34
120	3.87	6.21	9.97	16.00	25.68	41.22
110	2.54	4.08	6.55	10.51	16.86	27.07
100	1.28	2.02	3.25	5.21	8.36	13.42
90	0.00	0.00	0.00	0.00	0.00	0.00
80	−1.28	−2.02	−3.25	− 5.21	− 8.36	−13.42
70	−2.54	−4.08	−6.55	−10.51	−16.86	−27.07
60	−3.87	−6.21	−9.97	−16.00	−25.68	−41.22

The Tangential Velocity v in Meters per Second

$a z$ = 180°	0.00	0.00	0.00	0.00	0.00	0.00
170	3.00	4.82	7.73	12.41	19.92	31.97
160	4.21	6.76	10.85	17.42	27.95	44.86
150	5.09	8.17	13.12	21.06	33.79	54.24
140	5.78	9.27	14.88	23.88	38.32	61.50
130	6.31	10.12	16.24	26.07	41.84	67.14
120	6.70	10.76	17.27	27.72	44.48	71.39
110	6.98	11.21	17.99	28.87	46.34	74.36
100	7.15	11.47	18.42	29.55	47.43	76.13
90	7.20	11.56	18.56	29.78	47.80	76.71
80	7.15	11.47	18.42	29.55	47.43	76.13
70	6.98	11.21	17.99	28.87	46.34	74.36
60	6.70	10.76	17.27	27.72	44.48	71.39

The Vertical Velocity w in Meters per Second

as = 180°	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
170	0.0007	0.0018	0.0046	0.0120	0.0308	0.0794
160	0.0014	0.0036	0.0091	0.0236	0.0607	0.1564
150	0.0020	0.0052	0.0134	0.0345	0.0887	0.2286
140	0.0026	0.0067	0.0172	0.0443	0.1141	0.2939
130	0.0031	0.0080	0.0205	0.0528	0.1340	0.3502
120	0.0035	0.0090	0.0232	0.0597	0.1537	0.3959
110	0.0038	0.0098	0.0251	0.0648	0.1668	0.4296
100	0.0040	0.0102	0.0263	0.0679	0.1748	0.4502
90	0.0040	0.0104	0.0268	0.0689	0.1775	0.4572
80	0.0040	0.0102	0.0263	0.0679	0.1748	0.4502
70	0.0038	0.0098	0.0251	0.0648	0.1668	0.4296
60	0.0035	0.0090	0.0232	0.0597	0.1537	0.3959

The Ocean and the Land Cyclones

The tornadoes and hurricanes always occur in strata of air which are practically quiescent in the vertical direction, the tornadoes in the lower levels of stagnant air during hot summer afternoons, and the hurricanes in the neighborhood of the latitudes 30° to 35°, where the east and west movements in the general circulation practically disappear. Should hurricanes move into higher latitudes, where the eastward drift prevails with an increase of its velocity in proportion to the height above the ocean, the nearly perfect dumb-bell vortices which represent them are transformed into imperfect vortices of the same general type. The penetration of the head of the vortex into the midst of the eastward drift introduces components of resistance which deplete and even destroy the type in the upper levels, so that it is degraded to a cyclone, or imperfect dumb-bell vortex by the mere mechanical action. Furthermore, the temperature distribution is distinctly different in hurricanes and in cyclones. In the former the temperature differences are separated by horizontal planes in the upper levels, while in the latter the temperatures are separated chiefly in a vertical direction. The hurricanes have a symmetrical horizontal distribution of temperatures, but in cyclones the temperature distribution is decidedly asymmetrical, as is well known from the weather maps on the surface. The same asymmetry of temperature continues to the levels as high as 10,000 meters, warm on the east and cold on the west side of cyclones in the United States. These broad, thin sheets of warm and cold air, under the action of gravity, tend to return to a horizontal symmetry by the production of stream lines, whereby the cold air underruns the warm stream to the east and to the west by dividing into two branches, while the warm air overruns the cold air to the east and to the west similarly in two branches. Complicated stream lines are thus produced, which are those observed in the free air, after entering into composition with the velocities of the general circulation of the locality. This complex subject will require much more study than has been possible up to the present time in order to secure a complete analysis of the data, but it is clear that the research must proceed along certain lines which can be briefly indicated.

The first problem is to separate the imperfect from the perfect vortices, and to assign the components of resistance, that is, to construct a reverse vortex which is practically equivalent to the system of reactions that prevents the dumb-bell vortex from developing into a pure form. The second problem is to determine the stream lines by which the masses of air at different temperatures are drawn by the force of gravity into these imperfect cyclonic vortices. The ocean cyclone, October 11, 1905, has been taken to illustrate the composition of vortices, and the land cyclones must be studied more at length from the data provided by balloon and kite ascensions in Europe and the United States. The ocean cyclone is more highly developed than the land cyclone, and affords a convenient transition from the hurricane to the ordinary cyclonic storm. The cyclone of October 11, 1905, has been reduced to an equivalent cylindrical vortex by taking the mean radii as measured in four directions at right angles to each other. This mechanical process need not be repeated here, but the result is that the radii are not spaced in the vortical geometrical ratio. They diverge from that model which belongs to the perfect vortex. The corresponding velocities tangential to the equivalent circular isobars were constructed from the observed values in different parts of the cyclone as reported by the 110 vessels that made observations on that date.

TABLE 52

THE OCEAN CYCLONE, OCTOBER 11, 1905

The *Imperfect* Dumb-bell-shaped Vortex ψ_1

Results for the Plane $a\,z = 60°$

Initial $z = 8000$ meters, $a = \frac{180°}{8000 + 4000} = 0.015$, $\log \rho = 0.10600$.

z	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
A_1	.00089	.00127	.00184	.00265	.00382	.00551	.00795	.01146

The Radii ϖ in Kilometers

$a\,z = 180°$	∞	∞	∞	∞	∞	∞	∞	∞
170	3079.9	2412.9	1890.3	1480.6	1160.2	909.0	712.1	557.9
160	2194.6	1719.3	1346.9	1055.2	826.7	647.7	507.4	397.5
150	1815.0	1422.0	1114.0	872.8	683.7	535.7	419.6	328.8
140	1600.8	1254.1	982.5	769.7	603.0	472.4	370.1	290.0
130	1466.4	1148.8	900.0	705.1	552.4	432.8	339.0	265.6
120	1379.2	1080.5	846.5	663.2	519.5	407.0	318.9	249.8
110	1324.0	1037.2	812.6	636.6	498.8	390.7	306.1	239.8
100	1293.3	1013.2	793.8	621.9	487.2	381.7	299.0	234.3
90	1283.4	1005.5	787.7	617.1	483.5	378.8	296.7	232.5
80	1293.3	1013.2	793.8	621.9	487.2	381.7	299.0	234.3
70	1324.0	1037.2	812.6	636.6	498.8	390.7	306.1	239.8
60	1379.2	1080.5	846.5	663.2	519.5	407.0	318.9	249.8

The Radial Velocity u_1 in Meters per Second

$a\,z = 180°$	∞	∞	∞	∞	∞	∞	∞	∞
170	40.2	45.4	51.3	58.0	65.5	74.0	83.6	94.4
160	27.3	30.9	34.9	39.4	44.5	50.3	56.8	64.2
150	20.8	23.5	26.6	30.0	33.9	38.3	43.0	48.9
140	16.2	18.4	20.7	23.4	26.5	29.9	33.8	38.1
130	12.5	14.1	16.0	18.0	20.4	23.0	26.0	29.4
120	9.1	10.3	11.7	13.2	14.9	16.8	19.0	21.5
110	6.0	6.8	7.7	8.7	9.8	11.0	12.5	14.1
100	3.0	3.4	3.8	4.3	4.8	5.5	6.2	7.0
90	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
80	− 3.0	− 3.4	− 3.8	− 4.3	− 4.8	− 5.5	− 6.2	− 7.0
70	− 6.0	− 6.8	− 7.7	− 8.7	− 9.8	−11.0	−12.5	−14.1
60	− 9.1	−10.3	−11.7	−13.2	−14.9	−16.8	−19.0	−21.5

The Tangential Velocity v_1 in Meters per Second

$az = 180°$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
170	7.1	8.0	9.0	10.2	11.6	13.0	14.7	16.7
160	9.9	11.2	12.7	14.3	16.2	18.3	20.7	23.4
150	12.0	13.6	15.4	17.3	19.6	22.1	25.0	28.3
140	13.6	15.4	17.4	19.7	22.2	25.1	28.4	32.0
130	14.9	16.8	19.0	21.5	24.3	27.4	31.0	35.0
120	15.8	17.9	20.2	22.8	25.8	29.1	32.9	37.2
110	16.5	18.6	21.0	23.8	26.9	30.3	34.3	38.7
100	16.9	19.1	21.5	24.3	27.5	31.1	35.1	39.7
90	17.0	19.2	21.7	24.5	27.7	31.3	35.4	40.0
80	16.9	19.1	21.5	24.3	27.5	31.1	35.1	39.7
70	16.5	18.6	21.0	23.8	26.9	30.3	34.3	38.7
60	15.8	17.9	20.2	22.8	25.8	29.1	32.9	37.2

The Vertical Velocity w_1 in Meters per Second

$az = 180°$	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
170	.0003	.0004	.0006	.0009	.0013	.0019	.0028	.0040
160	.0006	.0009	.0013	.0018	.0026	.0038	.0054	.0078
150	.0009	.0013	.0018	.0026	.0038	.0055	.0079	.0115
140	.0011	.0016	.0024	.0034	.0049	.0071	.0102	.0147
130	.0014	.0020	.0028	.0041	.0059	.0084	.0122	.0176
120	.0015	.0022	.0032	.0046	.0066	.0095	.0138	.0198
110	.0017	.0024	.0035	.0050	.0072	.0104	.0149	.0215
100	.0017	.0025	.0036	.0052	.0075	.0109	.0157	.0226
90	.0018	.0025	.0037	.0053	.0076	.0110	.0159	.0229
80	.0017	.0025	.0036	.0052	.0075	.0109	.0157	.0226
70	.0017	.0024	.0035	.0050	.0072	.0104	.0149	.0215
60	.0015	.0022	.0032	.0046	.0066	.0095	.0138	.0198

TABLE 53

THE OCEAN CYCLONE, OCTOBER 11, 1905

The *Perfect* Dumb-bell-Shaped Vortex ψ_0

The Radii ϖ remain the Same

z	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
A_0	.00089	.00144	.00235	.00382	.00623	.01014	.01653	.02699

The Radial Velocity u_0

$az = 180°$	∞	∞	∞	∞	∞	∞	∞	∞
170	40.2	51.3	65.5	83.6	106.7	136.2	173.8	221.9
160	27.3	34.9	44.5	56.8	72.5	92.6	118.2	150.9
150	20.8	26.6	33.9	43.3	55.3	70.6	90.1	115.0
140	16.2	20.7	26.5	33.8	43.1	55.1	70.3	89.7
130	12.5	15.9	20.4	26.0	33.2	42.3	54.0	69.0
120	9.1	11.7	14.9	19.0	24.3	31.0	39.5	50.4
110	6.0	7.7	9.7	12.5	15.9	20.3	26.0	33.1
100	3.0	3.8	4.8	6.2	7.9	10.1	12.9	16.4
90	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

The Tangential Velocity v_0

az °								
az = 180	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
170	7.1	9.0	11.6	14.7	18.8	24.0	30.7	39.1
160	9.9	12.7	16.2	20.7	26.4	33.7	43.0	54.9
150	12.0	15.4	19.6	25.0	31.9	40.8	52.0	66.4
140	13.6	17.4	22.2	28.4	36.2	46.2	59.0	75.3
130	14.9	19.0	24.2	31.0	39.5	50.4	64.4	82.2
120	15.8	20.2	25.8	32.8	42.0	53.6	68.5	87.4
110	16.5	21.0	26.9	34.3	43.8	55.9	71.3	91.0
100	16.9	21.5	27.5	35.1	44.8	57.2	73.0	93.2
90	17.0	21.7	27.7	35.4	45.1	57.6	73.6	93.9

The Vertical Velocity w_0

az °								
az = 180	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
170	.0003	.0005	.0008	.0013	.0022	.0035	.0057	.0094
160	.0006	.0010	.0016	.0026	.0043	.0069	.0113	.0184
150	.0009	.0014	.0024	.0038	.0062	.0101	.0165	.0269
140	.0011	.0018	.0030	.0049	.0080	.0130	.0212	.0346
130	.0014	.0022	.0036	.0058	.0095	.0155	.0253	.0412
120	.0015	.0025	.0041	.0066	.0108	.0176	.0286	.0466
110	.0017	.0027	.0044	.0072	.0117	.0191	.0311	.0506
100	.0017	.0028	.0046	.0075	.0123	.0202	.0326	.0530
90	.0018	.0029	.0047	.0076	.0124	.0203	.0331	.0538

TABLE 54

THE REVERSING OR COMPONENT VORTEX, $\psi_1 - \psi_0 = \psi_2$

$A_2 = A_1 - A_0$	.00000	−.00017	−.00051	−.00117	−.00241	−.00463	−.00858	−.01553
The radial velocity of the reverse vortex, u_2								
az = 180°	0.0	....	....	....	....	....	....	....
170	0.0	−5.9	−14.2	−25.6	−41.2	−62.2	−90.2	−127.5
160	0.0	−4.0	− 9.6	−17.4	−28.0	−42.3	−61.4	− 86.7
150	0.0	−3.1	− 7.3	−13.3	−21.4	−32.3	−47.1	− 66.1
140	0.0	−2.3	− 5.8	−10.4	−16.6	−25.2	−36.5	− 51.6
130	0.0	−1.8	− 4.4	− 8.0	−12.8	−19.3	−28.0	− 39.6
120	0.0	−1.4	− 3.2	− 5.8	− 9.4	−14.2	−20.5	− 28.9
110	0.0	−0.9	− 2.0	− 3.8	− 6.1	− 9.3	−13.5	− 19.0
100	0.0	−0.4	− 1.0	− 1.9	− 3.1	− 4.6	− 6.7	− 9.4
90	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
The tangential velocity of the reverse vortex, v_2								
az = 180°	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
170	0.0	−1.0	−2.6	− 4.5	− 7.2	−11.0	−16.0	−22.4
160	0.0	−1.5	−3.5	− 6.4	−10.2	−15.4	−22.3	−31.5
150	0.0	−1.8	−4.2	− 7.7	−12.3	−18.7	−27.0	−38.1
140	0.0	−2.0	−4.8	− 8.7	−14.0	−21.1	−30.6	−43.3
130	0.0	−2.2	−5.2	− 9.5	−15.2	−23.0	−33.4	−47.2
120	0.0	−2.3	−5.6	−10.0	−16.2	−24.5	−35.6	−50.2
110	0.0	−2.4	−5.9	−10.5	−16.9	−25.6	−37.0	−52.3
100	0.0	−2.4	−6.0	−10.8	−17.3	−26.1	−37.9	−53.5
90	0.0	−2.5	−6.0	−10.9	−17.4	−26.3	−38.2	−53.9

The vertical velocity of the reverse vortex, w_2

az = 180°	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
170	.0000	−.0001	−.0002	−.0004	−.0009	−.0016	−.0029	−.0054
160	.0000	−.0001	−.0003	−.0008	−.0017	−.0031	−.0059	−.0106
150	.0000	−.0001	−.0005	−.0012	−.0024	−.0046	−.0086	−.0154
140	.0000	−.0002	−.0006	−.0015	−.0031	−.0059	−.0110	−.0199
130	.0000	−.0002	−.0008	−.0017	−.0036	−.0071	−.0131	−.0236
120	.0000	−.0003	−.0009	−.0020	−.0042	−.0081	−.0148	−.0268
110	.0000	−.0003	−.0009	−.0022	−.0045	−.0087	−.0162	−.0291
100	.0000	−.0003	−.0010	−.0023	−.0047	−.0091	−.0169	−.0301
90	.0000	−.0004	−.0010	−.0023	−.0048	−.0093	−.0172	−.0309

The Composition of Vortices

By subtracting the computed velocities of the perfect vortex from those of the imperfect vortex, $u_1 - u_0 = u_2$, $v_1 - v_0 = v_2$, $w_1 - w_0 = w_2$, we have a component vortex which, added to the perfect vortex, will produce the observed imperfect vortex, $u_1 = u_0 + u_2$, $v_1 = v_0 + v_2$, $w_1 = w_0 + w_2$, the signs being added algebraically. The corresponding values of the constants A, $A_1 - A_0 = A_2$, can be found by computing the values of A for the derived velocities by the formulas,

$$(643) \quad A_2 = \frac{u_2}{a\varpi \cos az} = \frac{v_2}{a\varpi \sin az} = \frac{-w_2}{2 \sin az}.$$

More simply, the algebraic values of A_2 are derived immediately from A_1 (imperfect vortex) − A_0 (perfect vortex), whence the corresponding velocities u_2, v_2, w_2 can be computed in the usual manner.

Table 52 gives the values of A_1, u_1, v_1, w_1 in the imperfect vortex; Table 53 those of A_0, u_0, v_0, w_0 in the perfect vortex, and Table 54 those of A_2, u_2, v_2, w_2 in the component reversing vortex. A comparison of the velocities in these tables shows that, by starting with the same radius and velocity on the outer isobar (1), the observed *imperfect* vortex departs more and more from the corresponding *perfect* vortex in proportion as the velocities approach the axis. The component vortex which is equivalent to these differences is a vortex *reversed* in all respects to the original vortex, revolving in the opposite direction and directed downward from the clouds to the surface of the sea. This principle of the composition of vortices through the constants A

of the successive tubes is very important, and leads to many practical researches in the theory of cyclones, because it enables us to take account of the numerous departures from the pure vortex law, without giving up the advantages of the method of vortex computations.

The Reversed Dumb-bell Vortex

A very erroneous impression would be left if it were supposed that the imperfect dumb-bell vortex could be applied directly to the study of the common cyclones in the atmosphere. The dumb-bell vortex seems to be essentially *reversed*, turned

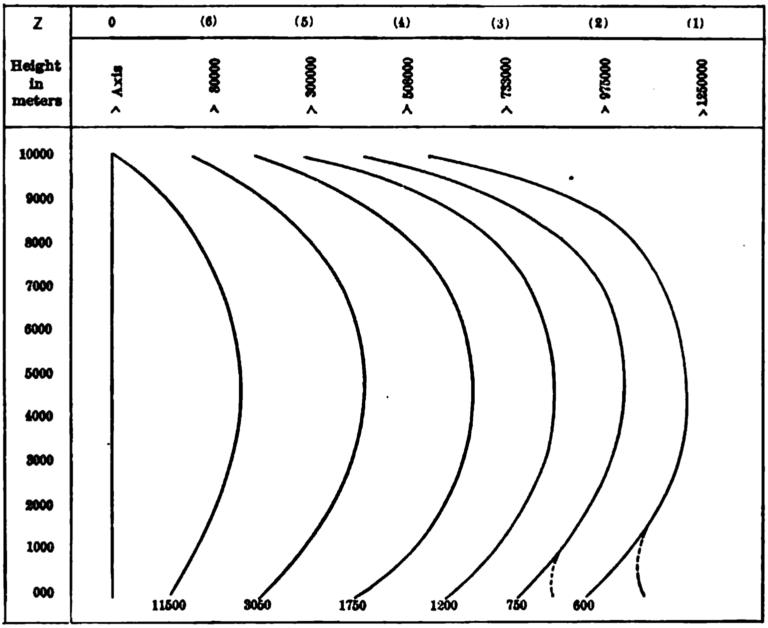


FIG. 27. The constant $a\,A$ lines as derived from observations in the land cyclones.

inside out, as can be seen by Table 55, and Fig. 27. The discussion of the velocity components resulted as given in the Cloud Report, in Table 126. Taking the velocities in that table and plotting them on diagrams, a consistent system was deduced which conforms on the average to that there given. It is produced on Table 55, I, II, for every 1,000-meter level, and for the radial distances given in the normal land cyclone. Since $\tan a\,z = \frac{v}{-u}$, by (527) and (528), we can compute $a\,A$ for each level and tube. The result appears in Table 55, III. On Fig. 27, at the radial distances, 80, 300, 508, 733, 975, 1,250 kilometers, these values of $a\,A$ were plotted down, and lines of equal $a\,A$ were drawn and they are given on Fig. 27. They contain the surprising result that the radial distances ϖ are arranged nearly on a geometrical ratio system, as can be readily seen by making the tests, and that the old ratio value $\log \rho = 0.20546$ is quite competent to satisfy the average conditions. The lines $a\,A$ are, however, *concave* towards the axis, instead of convex as in the hurricane, and the lines are closed up on the *outer* circles rather than on the inner, this being a complete *reversal* of the configuration. The theoretical and the thermodynamical conditions which produce this circulating structure have been indicated in the series of papers on the "Thermodynamics of the Atmosphere," W. B. 372, 1907. The subject will require further study and investigation.

TABLE 55

THE OBSERVED RADIAL AND TANGENTIAL VELOCITIES IN CYCLONES. INTERNATIONAL CLOUD REPORT

I. u = the observed radial velocities

Height in Meters	(1) 1250000	(2) 975000	(3) 733000	(4) 508000	(5) 300000	(6) 80000
10000	−0.5	−0.7	−1.0	−1.3	−1.6	−2.0
9000	−1.5	−2.1	−2.7	−3.0	−3.7	−4.5
8000	−2.0	−3.0	−4.0	−4.5	−5.5	−6.0
7000	−3.0	−4.0	−5.0	−6.0	−7.0	−7.5
6000	−3.0	−4.0	−5.0	−6.0	−7.0	−8.0
5000	−2.5	−3.0	−4.0	−4.5	−6.0	−7.0
4000	−2.0	−2.5	−3.0	−4.0	−5.0	−6.0
3000	−2.0	−2.0	−2.0	−2.5	−3.5	−4.5
2000	−2.0	−2.0	−2.0	−2.0	−2.0	−2.0
1000	−3.5	−3.5	−4.0	−4.0	−4.0	−4.0
000	−4.5	−5.0	−6.0	−6.5	−6.5	−6.0

II. v = the observed tangential velocities

10000	+2.0	+ 2.5	+ 3.0	+ 4.0	+ 4.5	+ 5.0
9000	+4.0	+ 6.0	+ 8.0	+ 8.5	+ 8.5	+ 9.0
8000	+4.0	+ 7.0	+10.0	+12.0	+12.0	+11.0
7000	+5.0	+10.0	+13.0	+14.0	+14.0	+14.0
6000	+6.0	+11.0	+14.0	+15.0	+15.0	+16.0
5000	+7.0	+12.0	+15.0	+16.0	+18.0	+19.0
4000	+7.0	+11.0	+14.0	+17.0	+19.0	+21.0
3000	+6.0	+10.0	+13.0	+16.0	+19.0	+23.0
2000	+5.0	+ 9.0	+12.0	+15.0	+18.0	+22.0
1000	+5.0	+ 6.0	+ 8.0	+10.0	+12.0	+14.0
000	+5.0	+ 5.5	+ 6.0	+ 6.0	+ 6.5	+ 7.0

III. The computed $a\,A = \frac{v}{\varpi \sin a z} = \frac{-u}{\varpi \cos a z}$

10000	165	266	431	828	1593	6732
9000	342	652	1152	1775	3091	12577
8000	358	781	1469	2523	4401	15659
7000	467	1105	1900	2998	5217	19852
6000	537	1201	2028	3180	5518	24801
5000	595	1269	2118	3270	6325	25306
4000	585	1157	1953	3439	6548	27298
3000	506	1046	1794	3187	6441	29300
2000	431	946	1659	2978	6040	27598
1000	488	713	1220	2120	4216	18198
000	538	762	1158	1741	3064	11524

The unit = .00000001 = 1×10^{-8}.

It is obvious that the velocities u, v can be computed from the formulas, knowing the values of the constants $a\,A$, or, on the other hand, the constants can be computed from the observed u, v velocities. The results of this computation, Table 55, Fig. 27, show that the dumb-bell vortex system has been entirely reversed. The $a\,A$ lines are *concave* toward the axis, they are geometrically spaced but closed up on the *outer* rather than on the inner areas of the cyclone. The temperature distribution conforms to this arrangement, and the cause is probably due to the penetration of the vortex of the lower strata into the rapidly moving drift of the upper strata.

Historical Review of the Three Leading Theories Regarding the Physical Causes of Cyclones and Anticyclones in the Earth's Atmosphere

In the *Astronomical and Astrophysical Journal*, January, 1894, the writer made a summary of the three most important general theories regarding the physical causes which generate the local storms, called cyclones, in the earth's atmosphere, and the following extracts from that review are sufficiently explicit for ordinary purposes. The three theories are: (1) *Ferrel's* warm-center and cold-center cyclones; (2) *Hann's* dynamic production of temperatures as found by observation; (3) *Bigelow's* asymmetric cyclone with warm and cold currents arranged in ridges, or streams of different densities, and driven into local cyclonic and anticyclonic circulations by the force of gravitation acting upon them. Ferrel had in mind for his cyclone the type of the general circulation of the atmosphere, and conceived that the same principles dominate both of them. The general circulation is described as a cold-center cyclone, with eastward movement from the pole to the high-pressure belt in latitude 33°, and a warm ring of westward movement in the tropics; the local cyclone is warm at the center and has right-handed rotation from the axis to a ridge of high pressure, outside of which a cold ring circulates in the anti-right-handed direction in the northern hemisphere, these directions of motion being reversed in the southern hemisphere. Ferrel's practical difficulty was to account for the originating heat energy in the central column of the cyclone, and this must precede any criticism of the circulation that depends upon it. He writes, "In the ordinary cyclonic disturbances of the atmosphere, the causes are similar to those in the general circulation but more local, and consist of a difference of density arising mostly from a difference of temperature between some central area and the external surrounding parts of the atmosphere." This dominant idea proved fatal to Ferrel's successful development of sound fundamental principles, and has greatly influenced many students to travel a road whose end has never been found. He was evidently unable to account satisfactorily for the energy implied in the temperature difference required to do the work observed in the motions of the cyclones and anticyclones. In his Coast and Geodetic Survey Report, p. 183, he remarks: "If for any reason there is kept up a continued interchange of air between the central and exterior part"; p. 201, "The condensation of aqueous vapor plays an important part in cyclonic disturbances, but is by no means either a primary or a principal cause of cyclones"; p. 239, "Rainfall is not essential to the formation of areas of low barometer, and is not the principal cause of their formation or of their progressive motion"; in Waldo's edition of Ferrel's "Hydrodynamics," p. 39, "The theory which attributes the whole of the barometrical oscillations to the rarefaction of the atmosphere produced by the condensation of vapor in the formation of clouds and rain cannot be maintained." However, being hard pushed to find a cause for his central area of high temperature in cyclones, he gradually weakened from this position, endorsed Espy's condensation theory of the development of latent heat by the formation of clouds and rain, and in the last year of his life could write in *Science*, December 19, 1890, "All this has been done in the condensation theory of cyclones, with results so satisfactory as scarcely to leave a doubt as to the truth of the whole theory." This was written in reply to Dr. Hann's revolt against the sufficiency of this cause to produce cyclones as observed, who took the ground that these local gyrations are only subordinate whirls in the general circulation, which depend upon the effects of the equatorial radiation only, and are independent of any local cause. Hann even went so far as to conclude, that "the actual motion of the atmosphere is not a product of the temperature (Ferrel's idea), but is in spite of it; the temperature is a product of the motion," *Science* May 30, 1890. Ferrel was loyal to the theory that temperature differences cause the motion always and everywhere, and Hann, in adopting the inverse proposition, has surely erred against first principles.

Ferrel's mechanical theory urged him to adopt a ridge of high pressure surrounding every cyclone, as indicated in formulas (471) to (478), but this is opposed by several fundamental conditions which show that it does not conform with modern observations. Another solution for the warm-center cyclone is given in equations (479) to (490), but this is equally opposed by the observed conditions. Both of these mathematical solutions have been practically abandoned, chiefly because there are, in fact, no warm-centered cyclones and no cold-centered anticyclones in existence. The hurricane probably has a warm-centered system of motion, but it is entirely different in structure. When a cold sheet of air overflows a warm sheet, the warm sheet flows outward radially from a central point in all directions, like the spokes on a wheel, and this outward movement in the high cloud levels drags behind it the vortex tube described in Fig. 26, and Tables 51. This, however, is entirely different from the temperature conditions of motion in cyclones and anticyclones.

Bigelow writes in the same paper of the *Astronomical and Astrophysical Journal*, "I must admit freely that I am unable to see in the daily weather maps that formation as fundamental which Ferrel and meteorologists generally assume to be the primary state. I propose to see in temperature differences, arranged in waves or ridges, the true cause of the observed pressures and the antecedent of the precipitation. It is therefore necessary to account for cold and warm temperature waves passing over the United States." "The passage of winds past each other in opposite directions tends toward local gyrations, which all drift eastward with the prevailing component in middle latitudes. All this simply depends upon the difference between the polar and the equatorial temperature, and is fully in accordance with the views of Ferrel and the latest expressions by Dr. Hann." "The formation of these low- and high-pressure areas is the result of the existence of the warm or cold sections of waves lying athwart the maximum crest. From first principles the warm and cold masses will be impelled toward each other, because of the action of gravitation on media of differing density. They will tend to encounter along or near the ridge of greatest temperature variation. Along the line of greatest temperature change, with cold air to the west and warm air to the east of it, the gyrating cyclone is formed, the couple existing from the system of causes thus described. Likewise, along the next ridge, with cold air to the east and warm air to the west, and often to the south of the maximum crest, the anticyclone is produced. A corollary remark is that the storm track along the north United States seems to be the effort of the general circulation to restore the permanent polar low-pressure belt which is interrupted by the continent. Another is that tornadoes and hurricanes are due to precisely the same cause, namely, the juxtaposition of masses of air having great temperature differences." The ideas were illustrated by the cyclone of November 16, 17, 1893.

The origin of these cold and warm waves, or ridges of different densities, has been discussed at great length in the International Cloud Report; and the streams from the cold north and warm south were called "leakage" currents, because these are in fact sporadic offshoots from the general circulation into middle latitudes. The warm currents in the United States are thrown off by the Atlantic center of action, from the Gulf of Mexico to the north; those upon southeastern Asia from the Pacific center of action; those upon northwestern United States from the same Pacific center of action, or else from the Arctic zone over British America; those upon northwest Europe, from the Atlantic center of action, or else from the Arctic circulation. The continents and the oceans react upon the general circulation in such a manner as greatly to disturb and distort its free operation, so that finally southerly currents prevail in certain regions and northerly currents are dominant in other regions. In the United States the southerly warm currents and the northerly cold currents encounter in long streams, flowing past each other in waves or ridges of density, and under the force of gravitation they are compelled to flow in cyclonic and anticyclonic circulations toward a thermal equilibrium. The exact mathematical conditions prevailing at every point have been indicated in Chapter II of this Treatise, and in that place, and in Bulletin No. 3, Argentine Meteorological Office, 1912, the practical details have been illustrated. Bigelow, in 1894, laid down the principle of the force of gravitation acting upon density masses in alternate juxtaposition, whether side by side as in cyclones, anticyclones, and tornadoes, or in vertical superposition as in hurricanes, with intermediate cases between these principal positions; and in 1912 he worked out a method of computing the very important terms of the radiation energy in the general equations of motion, which had heretofore been entirely omitted from the discussions. A very brief summary of some of the most important features of the general and the local circulations are added in this place, though the student must consult the weather maps of various countries for any complete knowledge of such a complex subject as the actual circulation.

The General and Local Components of the Velocities, Pressures, and Temperatures in the Circulation of the Atmosphere

The General and the Local Components

There are certain distributions of the temperature, pressure, wind direction, and velocity, which are characteristic of the general motions of the atmosphere, and others which belong to the local circulations peculiar to the cyclones and anticyclones. It is necessary to separate them from the observed values that are the resultants of those two components. It is practicable to observe certain values of the velocities, pressures, and temperatures at a given station, from which the general or normal values are computed, so that by subtracting in the form of vectors the normal values from the observed values, the local or component terms may be found. This great labor has been performed for the United States and the West Indies, the results for the velocities being recorded in the Cloud Report, 1898, for the pressures in the Barometry Report, 1901, and for the temperatures in the Report on Homogeneous Normals, 1909, together with numerous papers in the *Monthly Weather Review*. This Treatise is concerned with the methods of computation and discussion appropriate to meteorology, rather than with statistical results, so that only a brief summary of these data can be presented in this connection. Unfortunately, the data of meteorology are so bulky that it becomes very difficult to do justice to the subject within the limits of a reasonable volume.

The Normal and Local Velocities in Storms

In the Cloud Report are to be found the resultant velocities and directions of the wind at the surface, in the cumulus levels (1,000–2,000 meters), and in the cirrus levels (8,000–10,000 meters), for all parts of cyclones and anticyclones, when the centers of these areas are located in different parts of the United States, as the Dakotas, the Lake region, New England, Colorado, Texas, and East Gulf States, respectively. They were obtained in the several areas by making a composite chart from about

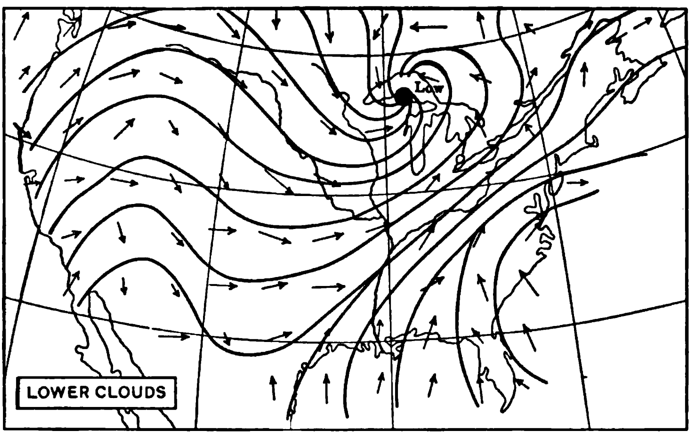


FIG. 28. Observed stream lines of air in the cumulus cloud level (2,000) over a cyclone whose center is in the Lake Region.

fifty charts for each type of storms. For this purpose the United States was divided into small areas by the parallels of latitude and the meridians, the centers of the fifty storms were made to coincide, and the vectors or arrows were transferred to a common chart, from which the resultant vector was carefully computed. These charts are of great theoretical value for the student, as well as of practical value for the forecaster, and they should be thoroughly examined. Specimens of these charts are given on Fig. 28 for a cyclone centered in the lake region, transcribing the

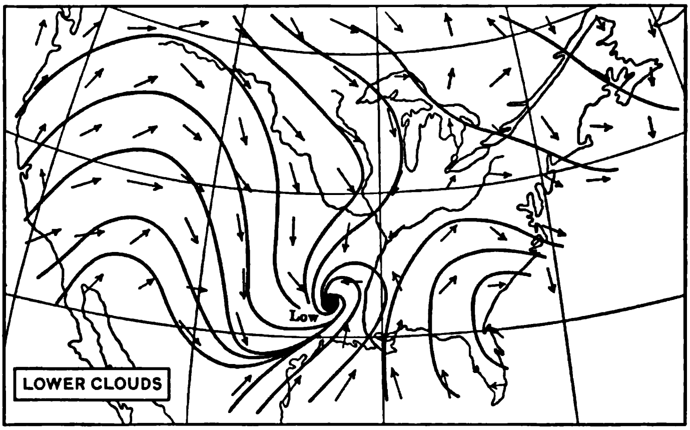


FIG. 29. Observed stream lines in the cumulus cloud level (2,000) over a cyclone whose center is in the West Gulf States.

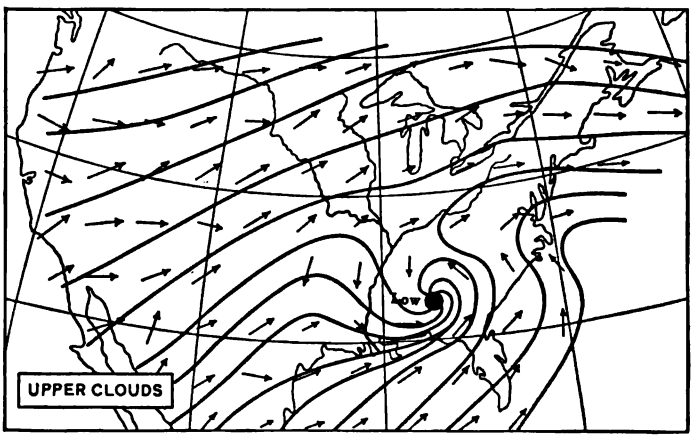


FIG. 30. Observed stream lines in the cirrus cloud level (10,000) over a hurricane whose center is in the South Atlantic States. These currents all show that there is a U-shaped formation in the circulation generally opening to the northeastward, though it is also found pointing westward and southward.

lower cloud level; on Fig. 29 for a cyclone in the West Gulf States for the cumulus cloud level; and on Fig. 30 for a hurricane in the East Gulf States, upon the upper or cirrus cloud level. It is seen in all these cases that the currents of air form U-shaped figures like the isobars and isotherms, in entering a cyclonic vortex, and that the eastward drift is locally diverted into this configuration. There is a bridge across the top of the U-shaped vortex, and in the isobars a well-defined saddle is always constructed, where the high-pressure areas are temporarily broken through in the construction of a vortex circulation.

Fig. 31 gives a representation of a typical circulation of air in connection with the isobars in three levels: sea level, 3,500-foot level, and 10,000-foot level, showing the relation of the currents to the isobars. The high-pressure cusps tend to approach over a bridge or saddle at *C S C*, the pressure being lower to the north and to the south of it. The number of the closed isobars decreases with the height, and it is usual for them to disappear at the level of 3,000–4,000 meters, and sometimes even lower. This is a proof that the dumb-bell vortex which dominates in hurricanes has almost entirely vanished in cyclones except in the lowest levels, the top being entirely depleted in the higher levels. This throws back the theory of cyclones into quite a different category of imperfect vortices and, considering the asymmetrical distribution of the temperature, it is evident that the currents are due to pressure gradients in the thin sheets of air of different temperatures in the process of mixing in the middle latitudes. The tendency for the currents to divide into two streams mutually underrunning and overrunning each other should be carefully noted.

In the Cloud Report are contained the data from which the average heights were computed where the several clouds are formed, and the average velocities were deduced at which they move nearly eastward, after the cyclonic and the anticyclonic components have been eliminated. Table 56 contains a summary of the velocities in high and low areas, the northward and southward components over high and low areas, the northward and southward components between the centers in the warm and cold streams, and the seasonal velocities. See Fig. 32.

The cloud forms, stratus, cumulus, strato-cumulus, alto-cumulus, alto-stratus, cirro-cumulus, cirro-stratus, cirrus, occur at certain well-defined heights on the average, and it is found by observation that they drift over the earth's surface at certain average velocities as shown on the diagram. This increase of velocity upward, from 7 meters per second at the surface to 40

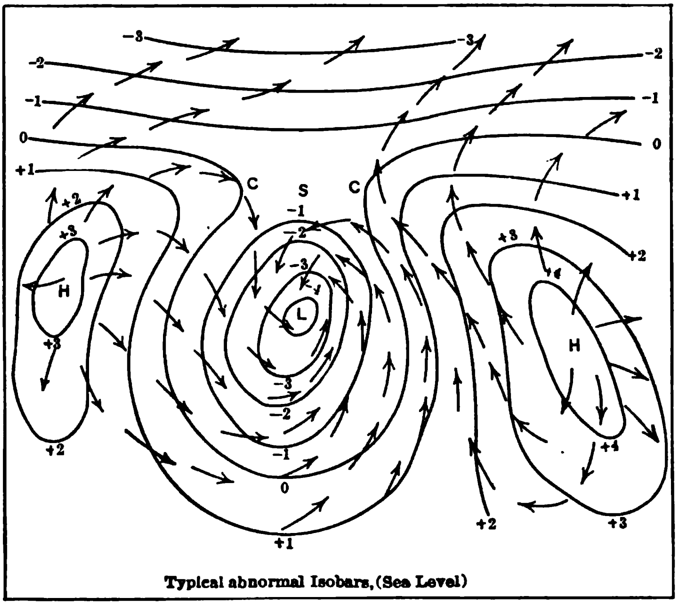
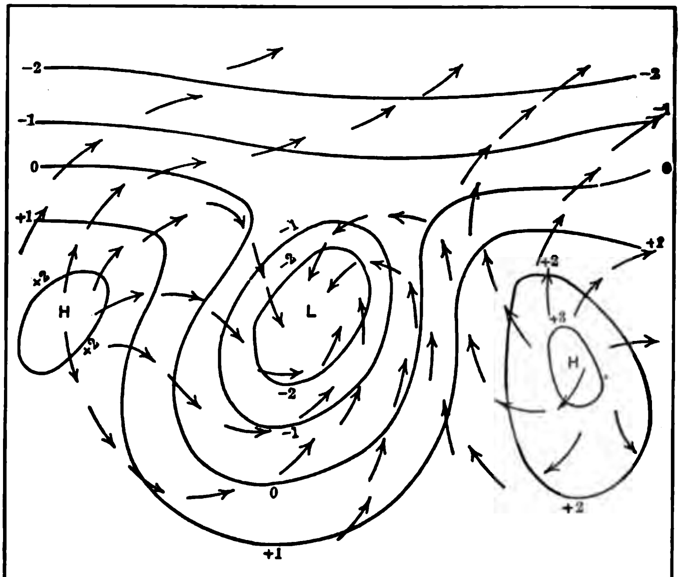


FIG. 31*a*.

FIG. 31*a*, *b*, *c*. Showing the relation of the local circulation to the typical isobars in high or low areas of pressure. The closed isobars form a rough vortex, which is supplied by the two-branched stream-lines and gradually dies out in the higher levels.

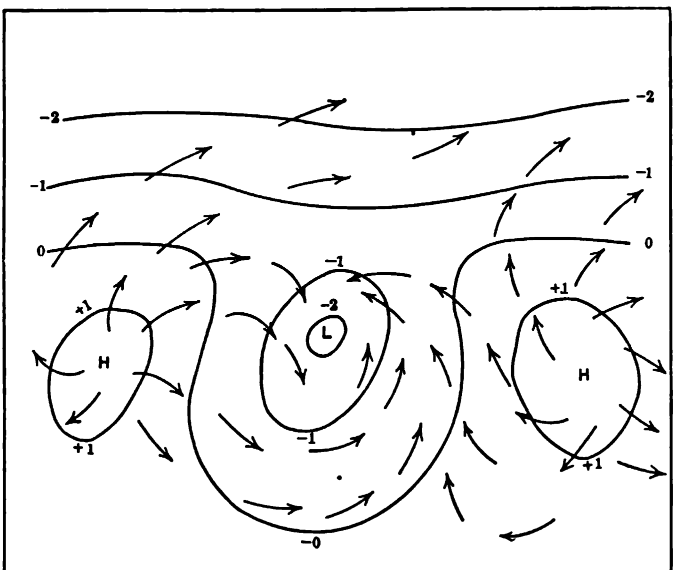
meters per second, is called the eastward drift in middle latitudes. This is the normal velocity component which must be eliminated from the observed component to produce the local disturbing component of velocity due to the cyclones and anticyclones proper.

The eastward and the westward drifts, in the middle latitudes and the tropics, respectively, are shown on Figs. 33, 34, 35.



Typical abnormal Isobars, (8,500 foot.)

FIG. 31*b*.



Typical abnormal Isobars, (10,000 foot)

FIG. 31*c*.

TABLE 56

SUMMARY OF THE VELOCITIES OF THE MOTIONS OF CLOUDS IN THE DIFFERENT LEVELS FOR THE MIDDLE ATLANTIC STATES. THE UPPER CLOUDS INCLUDE Ci., Ci.S., Ci.Cu., A.S., A.Cu., WITH THE MEAN HEIGHT 8.4 KILOMETERS, AND THE LOWER CLOUDS INCLUDE S.Cu., Cu., S., AT THE MEAN HEIGHT 2.4 KILOMETERS.

Symbols: Ci. = Cirrus.—S. = Stratus.—Cu. = Cumulus.—A. = Alto.

I. TOTAL VELOCITY IN HIGHS AND LOWS WITHOUT REGARD TO DIRECTIONS

Clouds	Ci.	Ci.S.	Ci.Cu.	A.S.	A.Cu.	S.Cu.	Cu.	S.	Wind	
Height in kilometers	9.8	9.8	8.1	5.9	4.5	2.5	1.5	0.9	0	
High Areas										*Per cent*
Total motion	34.9	39.1	33.5	30.2	23.5	23.3	11.2	11.4	4.8	....
Northern	38.3	42.6	33.9	31.1	26.6	22.7	10.9	12.2	4.9	....
Southern	30.4	34.8	30.5	24.1	19.7	18.5	10.4	9.5	4.8	19
Low Areas										
Total motion	40.8	39.8	39.3	36.0	29.2	28.6	14.6	11.1	5.4	15
Northern	44.6	42.5	43.8	39.4	32.6	32.9	17.4	13.2	5.3	....
Southern	28.3	36.3	34.8	30.5	24.4	21.1	11.8	8.6	5.9	28

II. SOUTHWARD AND EASTWARD COMPONENTS OF VELOCITIES IN HIGHS AND LOWS

Clouds	Ci.	Ci.S.	Ci.Cu.	A.S.	A.Cu.	S.Cu.	Cu.	S.	Wind	
High Areas										
+S −N	+ 1.97	+ 1.65	− 0.60	− 0.37	− 0.07	− 0.32	− 0.18	− 1.22	− 0.69	....
+E −W	+33.7	+32.0	+32.6	+27.2	+22.1	+16.0	+ 5.1	+ 5.8	+ 1.1	....
Low Areas										
+S −N	− 5.26	− 9.24	− 3.00	− 4.60	− 2.38	− 4.00	− 0.11	− 1.32	− 0.40	....
+E −W	+39.4	+35.9	+37.2	+31.3	+24.3	+24.3	+11.4	+ 7.3	+ 1.5	....

III. MEAN NORMAL COMPONENTS OF VELOCITY FOR THE UNITED STATES

Northward	− 1.6	− 3.3	− 1.3	− 2.5	− 1.2	− 2.2	− 0.1	− 1.3	− 0.5	....
Eastward	+36.6	+34.0	+34.9	+29.2	+23.2	+20.2	+ 3.3	+ 6.8	+ 1.3	....

IV. COMPONENT VELOCITIES IN SELECTED AREAS BETWEEN HIGH AND LOW CENTERS

Selected Areas Southward	+ 0.66	− 2.11	+ 4.95	+2.79	+ 6.24	+10.22	+ 6.52	+ 5.25	+ 2.23	....
	+40.1	+36.9	+33.7	+26.5	+23.7	+22.1	+ 9.6	+ 7.5	+ 3.2	....
Selected Areas Northward	− 3.75	− 3.89	− 7.34	− 7.47	− 7.78	−11.13	− 3.13	− 7.97	− 3.25	....
	+32.7	+36.9	+32.1	+31.0	+21.9	+17.1	+ 6.5	+ 5.1	+ 0.2	....

V. SEASONABLE VELOCITIES OF THE UPPER AND LOWER CLOUDS

Clouds	Upper clouds 8.4					Lower clouds 2.4				
Seasons	June	Sept.	Dec.	March	Ann.	June	Sept.	Dec.	March	Ann.
High Areas										
Northern	30.3	33.7	37.5	41.7	35.5	12.8	23.2	24.9	21.8	20.0
Southern	24.2	35.6	36.0	27.7	29.1	16.3	17.8	20.0	18.2	17.2
Low Areas										
Northern	39.7	45.2	47.4	37.1	42.6	19.2	32.8	31.5	27.5	27.7
Southern	25.9	30.9	33.1	34.8	32.4	12.9	21.9	18.7	16.1	19.6

All velocities in meters per second (1 m.p.s. = 2.2 miles per hour)

The scale on Fig. 35 is twice as great as on Fig. 33 or Fig. 34. It is seen that at San Juan, and generally in the West Indies, the westward drift in the lower levels reverses into an eastward drift in the upper levels, the transition occurring in the A.Cu. and A.St. levels. Hence, the westward trade winds are shallow,

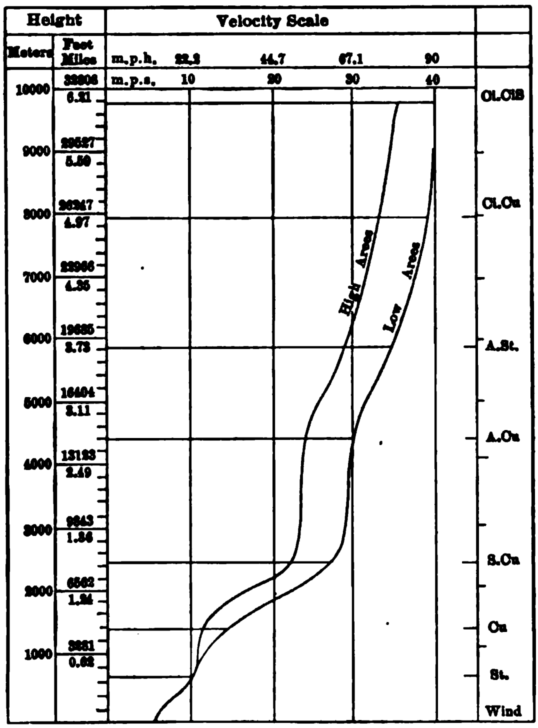


FIG. 32. The total eastward velocity in high and low areas. Cloud heights.

the greatest velocity westward being in the S.Cu. level (2,000–3,000 meters), and that they give way to the eastward drift which prevails over the subtropics in the higher levels. At Key West, in the midst of the North American high-pressure belt, the eastward and the westward drift is small in velocity, a similar reversal taking place in the middle levels. In the middle latitudes of the United States the eastward drift prevails in all levels, the velocities increasing from the surface upward. On these diagrams *J* stands for January and *D* for December, and all the intermediate months of the year are given in the line of

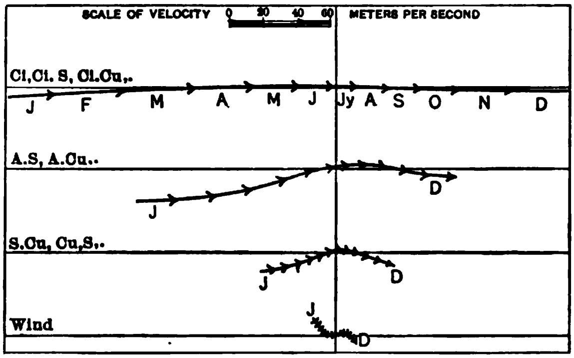


FIG. 33. The eastward drift above Washington, D. C., for each month in the year.

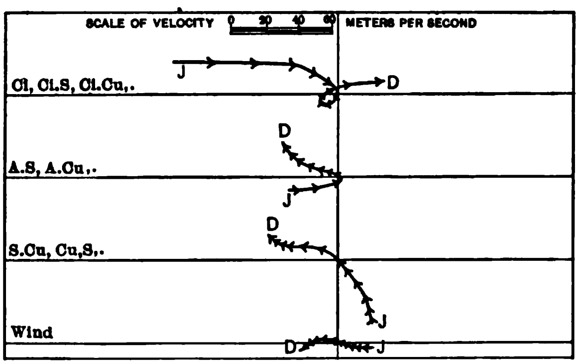


FIG. 34. The eastward and the westward drift above Key West, Florida.

vectors. In summer the velocities in the tropics for the upper levels are small and disturbed in direction, showing that the circulation is diminished when the sun is north of the equator. In all cases circulation depends upon contrasts in temperature, so that a vigorous circulation occurs in winter rather than in summer, when the temperatures of the air in the northern hemisphere are much more nearly equal than they are in winter.

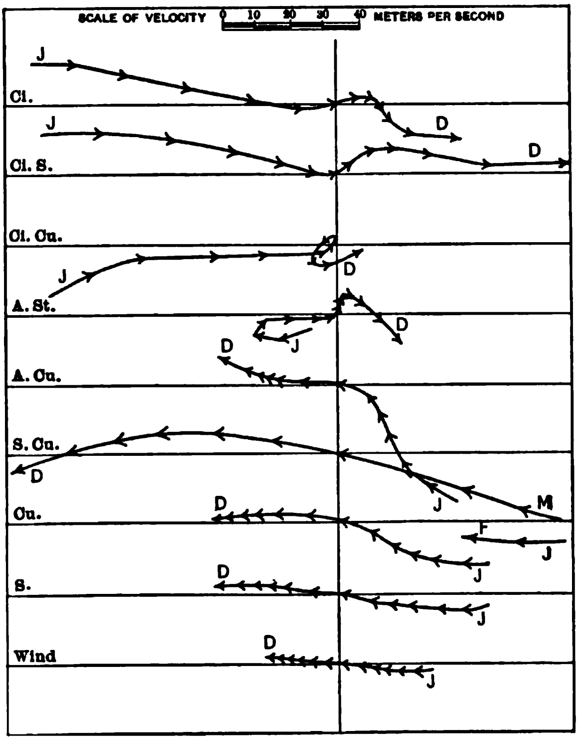


FIG. 35. The westward drift in the lower levels at San Juan, W. I., reversing into the eastward drift in the A.Cu., A.St. levels and the eastward drift in the upper levels.

The Normal and the Local Isobars in Cyclones and Anticyclones

The Analytic Construction of the Resultants

The general theory of the separation of components can be illustrated by the following figures.

Draw circles about the pole (see Fig. 36) representing barometric pressures at some level above the surface, ranging from 25.4 inches near the pole to 27.2 in latitude 20°. At two points superpose a series of local circles, 1, 2–8, representing a defect of pressure at (1), and an excess of pressure at (4). Add the respective values together at every point of intersection, and connect up the pressures having the same isobaric value. The resultant lines for (1) are seen at (2), and for (4) they are seen at (5).

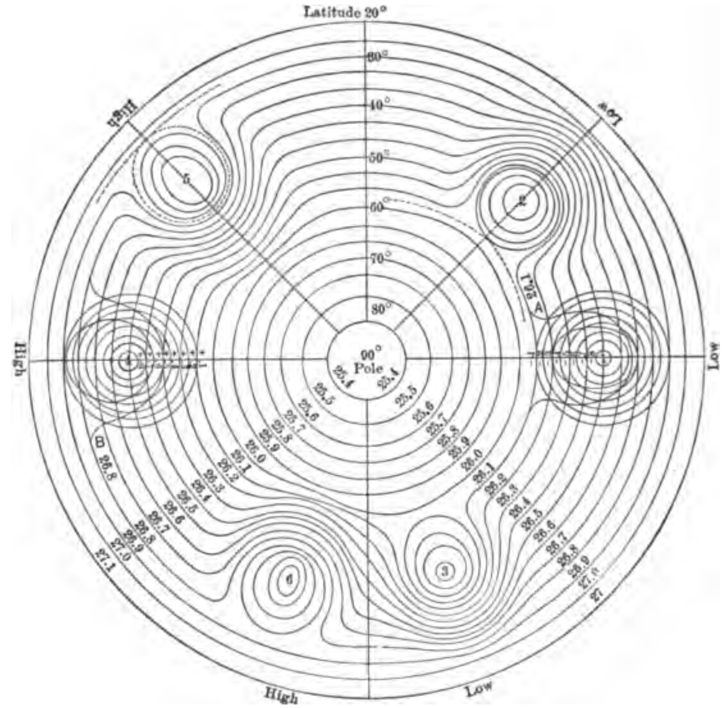


FIG. 36. The formation of cyclones in the general circulation about the poles of the earth.

Putting (2) and (5) in continuous figures the resultant disturbed values are found at (3) and (6), the ordinary form of the isobars observed in the atmosphere somewhat above the surface, as on the 3,500-foot and the 10,000-foot planes. The same facts can be determined analytically as illustrated by Fig. 37.

Take the co-ordinate systems as shown on the diagram.

Let R = the radius of the circle, $(a. b)$ the co-ordinates of the center, $(x. y)$ the co-ordinates of any point on the circle.

The general equation of the circle is

(644) $$(x - a)^2 + (y - b)^2 = R^2.$$

Take $b = 0$ and transpose the terms, so that,

(645) $$y^2 = -x^2 + 2ax + R^2 - a^2.$$

The equation of condition for the isobar which is the resultant of successive circular isobars added to successive straight-line

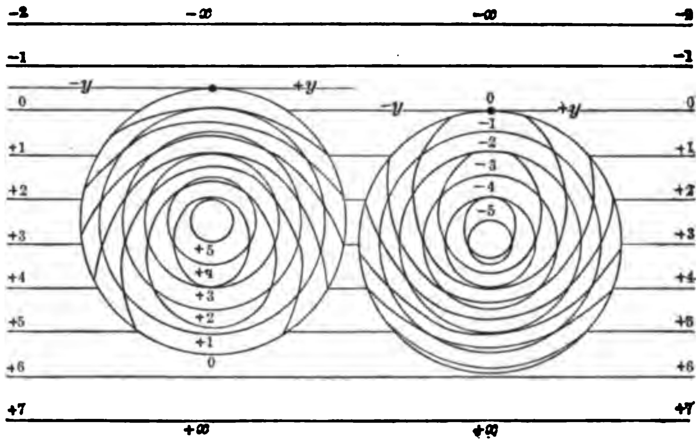


FIG. 37. The composition of right lines and circles where the gradients are twice as great on the lines as on the circles.

normal isobars, is that the sum of certain pair numbers shall be constant on the same line. Thus, $A + B =$ constant, where $A = nx$, some multiple of the ordinate x, and $B =$ the gradient number on the circles. For example, take the gradient on the normal straight lines one-half that on the normal circles, so that $n = \frac{1}{2}$, which is about the average in highly developed storms. Take successive circles, $R = 6, 5, 4, 3, 2$, whose gradient numbers are respectively $B = 0, -1, -2, -3, -4$. Take $a = 6$, $A = \frac{1}{2}x$, $A + B = 0$ for the 0-line, and $n = \frac{1}{2}$.

Similarly, by taking the proper groups of R, B, x for the $-1, +1 \ldots -2, +2, \ldots$ lines in low and high areas, we obtain the resultants shown on Fig. 37. Curves can be compounded analytically through their equations, when the equations of the lines are known, but as this is not usually the fact for the lines representing meteorological gradients, and lines of equal meteorological values, resort must be had to some graphic process of construction.

Graphic Construction of Resultants

The first step is to determine the normal isobars, isotherms, and wind vectors on certain selected planes. For the purpose we have chosen, as appropriate to forecasting requirements in the United States, the 3,500-foot and the 10,000-foot planes, besides the usual sea-level plane; and the Barometry Report contains the normals of pressure, temperature, and vapor pressure on these three planes, while the Cloud Report contains the normal vectors for the velocities of the air motions. We have, therefore, to reduce the observed data to these three planes, and subtract graphically the normals from them to obtain the local disturbing terms of the given element. Thus, in Fig. 38, if we obtained the oval lines from observation and subtracted the right lines from them, we should recover the circles or local components. Practically, lay down the normal values of the isobars or isotherms on a given plane, using transparent paper, superpose this upon the chart of observed values on that plane, and draw the diagonals of the quadrilateral figures that cover the diagram. In this way the charts of barometric pressure for the year 1903 on three planes (of which an example is given in Fig. 38) have been decomposed into the elements of the general and the local circulations from which we can study the general circulations on the one hand, and the local circulations on the other hand, without confusion.

The observed pressures at the stations of the United States can be reduced to these planes by means of suitable tables, so that they are easily embodied in a telegraphic report without delay to the forecast message. This was done for a few weeks in a preliminary study. It is to be noted that the closed isobars of the lower planes soon expand into loops in the upper level.

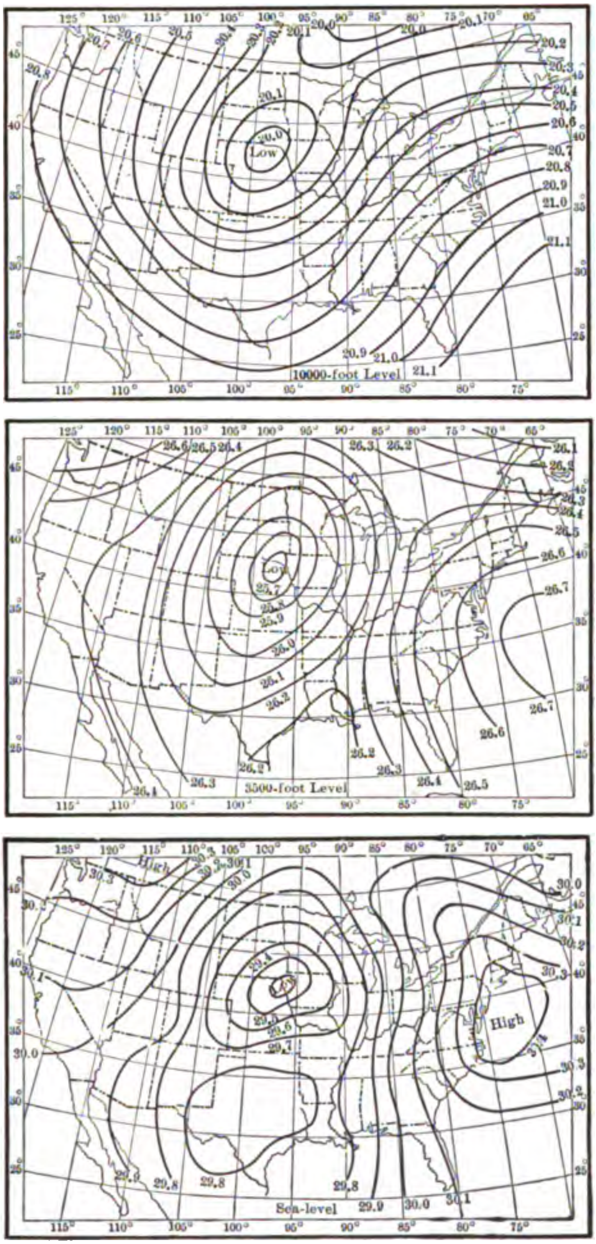


Fig. 38. The systems of isobars on three planes for the storm of February 27, 1903.

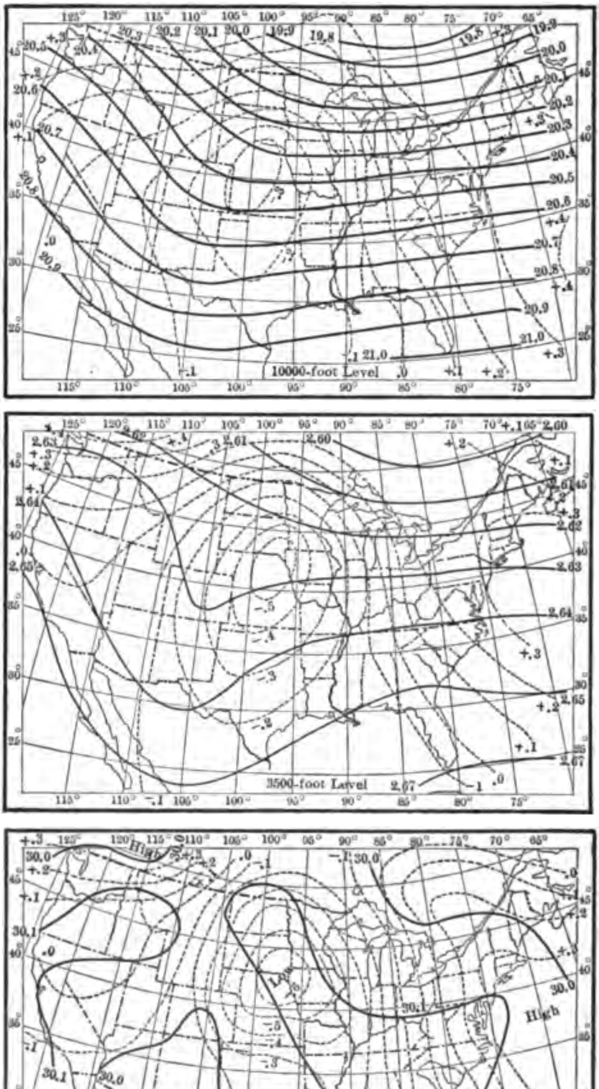


Fig. 39. The normal isobars (continuous lines) and the disturbing local isobars (dotted lines) in the storm of February 27, 1903. The normal lines were laid upon the observed lines of Fig. 38, and the dotted diagonals of Fig. 39 drawn. There are five closed isobars on the sea-level plane and only two closed isobars on the 10,000-foot plane. The system of high and low areas on the sea-level charts soon opens up into sinuous lines in the upper levels. A general view of this fact is given on Fig. 40.

The Normal and the Local Isotherms in Cyclones and Anticyclones

The isotherms as observed at a given time in the United States are separated by the same process into the two components, the normal and the local disturbance isotherms. Lay a chart containing the normal isotherms over the observed map and draw diagonal lines connecting up points having the same

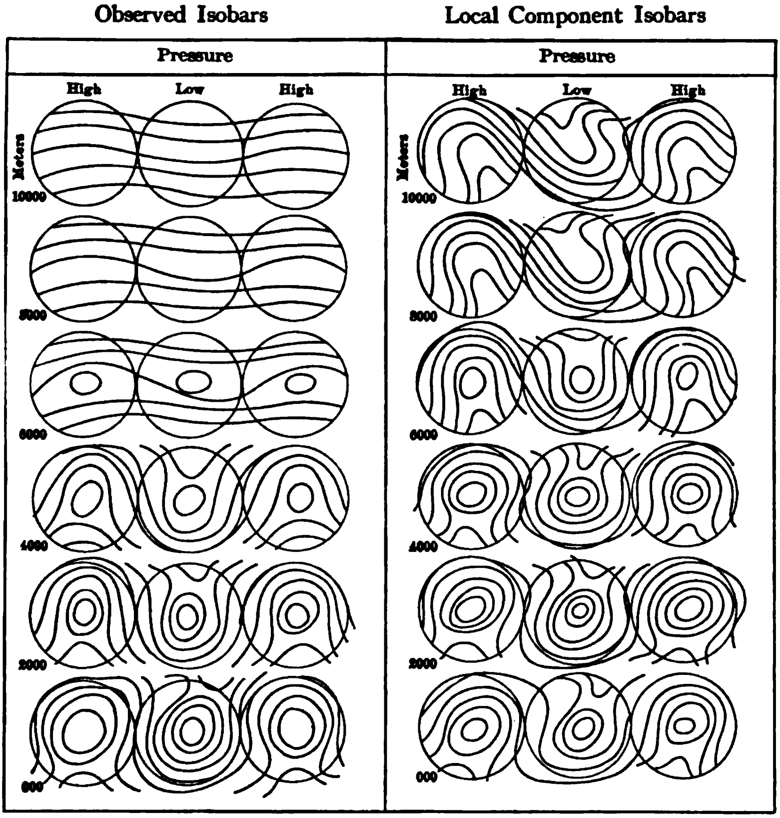


FIG. 40. Scheme of the distribution of the pressures in high and in low areas, in the observed and in the component isobars, on the levels up to 10,000 meters. These isobars are somewhat ideal, but they conform to conditions existing up to the top of the local disturbances in the atmosphere, that is to the cirrus region. The winter storms are cut off at 6,000 meters and even lower, while the summer storms can be traced much higher, on account of the relative retreat of the low temperatures to the higher levels. The U-shaped loops of the high areas open southward, and those of the low area open northward, so that in the upper levels there are sinuous, not closed isobars. Progress in forecasting consists in studying these upper plane auxiliary charts, in connection with the corresponding sea-level charts.

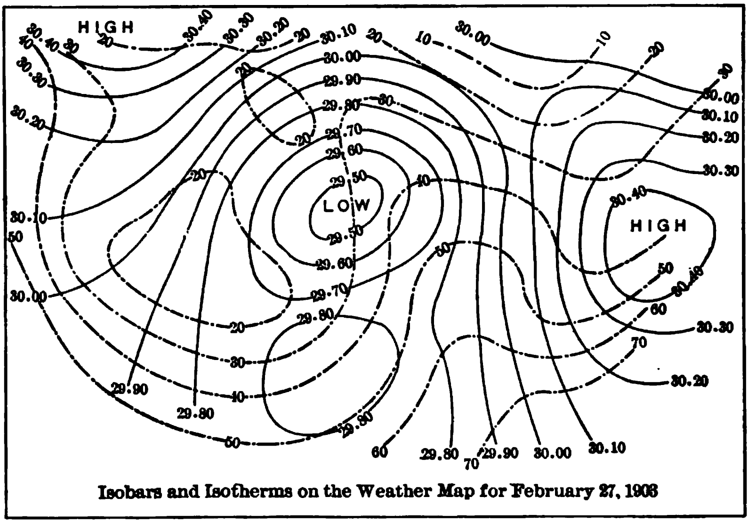


FIG. 41. The weather map of February 27, 1903, showing the observed isobars and isotherms.

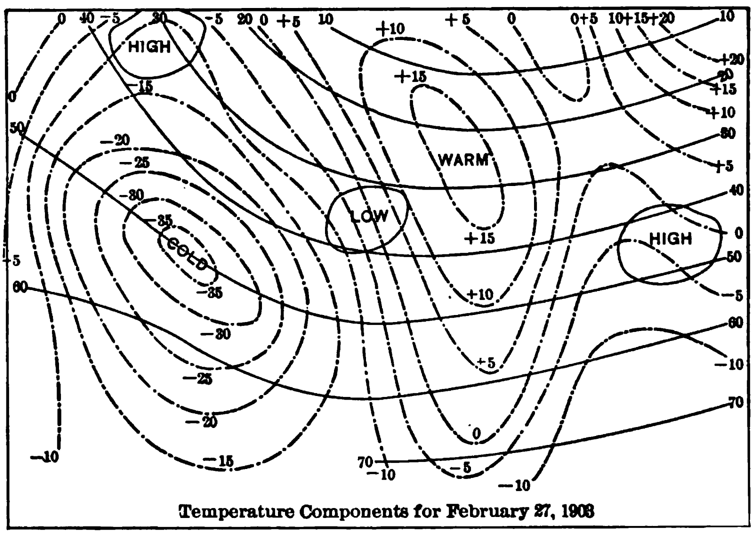


FIG. 42. The normal isotherms (full) and the local disturbance isotherms (dotted) which added together produce the observed isotherms of Fig. 41, Fahrenheit degrees of temperature.

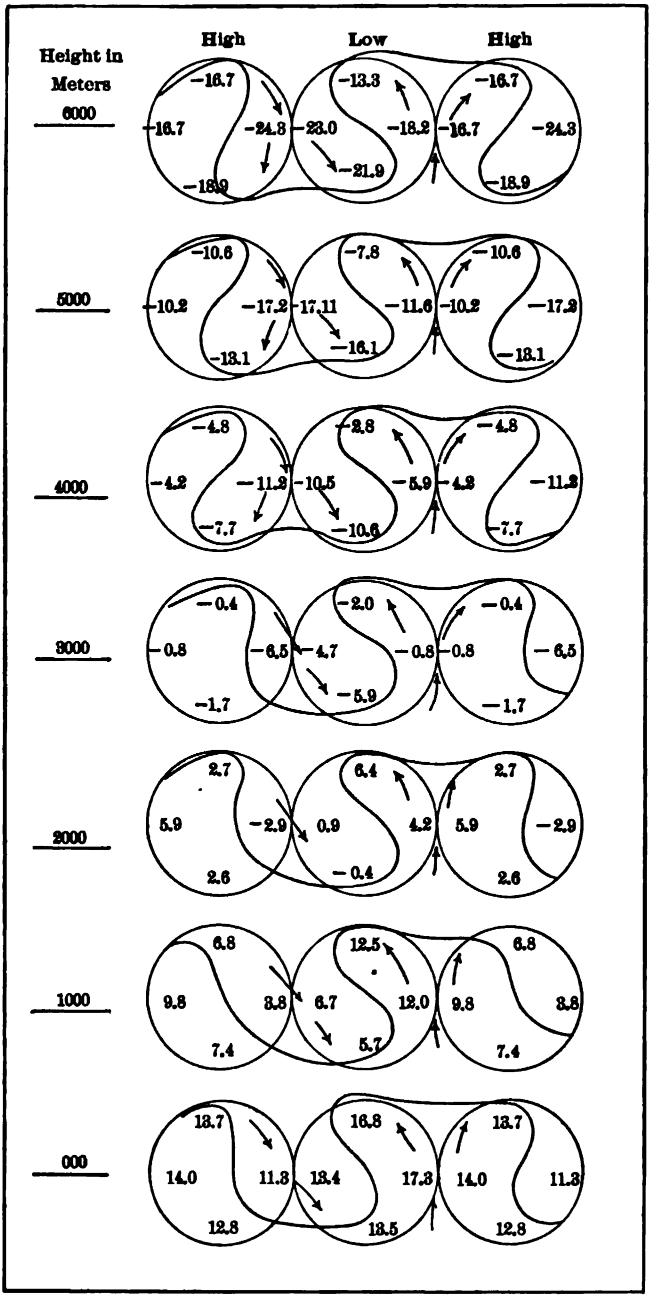


FIG. 43. Distribution of the high and low temperatures in cyclones and anticyclones up to the height of 6,000 meters, showing the tendency to divide into two branches with the maximum departure near the border of the high- and low-pressure areas. Centigrade degrees of temperatures.

difference of temperature, which are the local disturbing isotherms. Such normal charts of temperature on three planes are found in the Barometry Report. On Fig. 48 the composite of such a disturbance temperature system is given for nine cyclones, similar to that of February 27, 1903 (Fig. 41). Fig. 42 shows how the disturbance isotherms cover adjacent high- and low-pressure areas. These cold and warm areas, as distinguished from the normal temperatures of the season and place, are what accompany all anticyclonic and cyclonic disturbances. The wind currents are simply the effect of the force of gravity transporting these masses of air of different temperatures, so that the cold mass underruns the warm masses, and the warm mass overruns the cold masses to the right and the left hand, on all the levels simultaneously from the surface to the top of the disturbance. The campaign of extending the temperature observations into the higher levels is going on in different parts of the world, but definitive results have not been reached. There is no very general and fixed system of temperature values to be expected, because the incessant circulation, due to the annual change in declination of the sun, prevents the atmosphere from settling down into a simple thermal equilibrium. Fig. 43 gives an example of the distribution of the warm and cold areas in cyclones and anticyclones up to 6,000 meters. There is a tendency for the warm area to divide into two branches to the northward, and for the cold area to divide into two branches to the southward. The maximum departure of the temperature is somewhere between the centers of the low- and high-pressure areas, and it is not distributed symmetrically about the center as was assumed by Ferrel, and by the early German meteorologists, in the construction of their theories of vortex motion. This asymmetric theory of vortices was first discussed in 1894 and in the Cloud Report of 1898, and the defects of the other theories were pointed out. The pressure and temperature data of observation were then entirely lacking in the upper strata, so that several years were allowed to elapse before the subject could be properly resumed, as was done in 1906 in the series of papers on the "Thermodynamics of the Atmosphere," *Monthly*

Weather Review, and continued in 1907, 1908, in the series of papers on the vortices in the atmosphere. The labor of securing

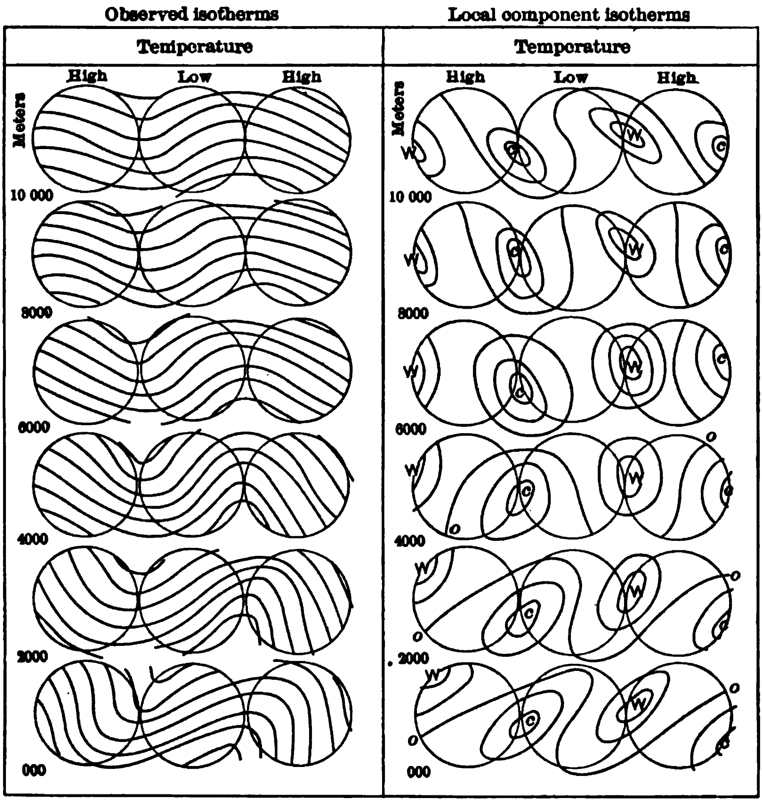


FIG. 44. The observed isotherms and the local disturbing isotherms in high and low areas of pressure from the surface to 10,000 meters. There is a tendency for the warm mass to ascend and rotate through about one quadrant, changing the direction of the horizontal axis from N.E. to N.W., and for the cold mass to descend and rotate through one quadrant changing the direction of the axis from the S.E. to the S.W. The sinuous lines in the upper levels deepen in the lower levels, chiefly because the rapid eastward drift in the upper levels, which smooths out all kinds of disturbances, relaxes in the lower levels, and permits the disturbance components to dominate more fully. Compare these diagrams with the Figs. 41, 42, 48, and note the position of the line of 0-departure. Further observations will improve the accuracy of these diagrams.

a sufficiently large amount of data in the upper levels, by balloon and kite ascensions, in order to eliminate temporary local conditions and secure average values, is so great that many years must elapse before meteorology will possess data for computations of precision in this field of research. Meanwhile it is very important for students to have in mind a picture of the general phenomena so that useless discussions may be avoided. It is particularly necessary that the temperature values due to the location of the observatory, as a hill or mountain, should be thoroughly eliminated, because the temperatures on an elevation of land are not the same as that in the free air at the same height. Otherwise, errors at the base station would go into the entire series of gradients through a mistaken process of computation.

The Normal and the Local Velocity Vectors in Cyclones and Anticyclones

The results of the observations made on the velocities of the cloud motions at Washington, D. C., are embodied in Fig. 45 for the anticyclones, in Fig. 46 for the cyclones, and in Fig. 47 the entire system of stream lines is laid down as a whole. The complete vector as observed in different areas surrounding the center is first plotted, and then the component vector after the mean eastward drift has been eliminated. The resultant vectors are long in the upper levels and short in the lower levels; the currents are slightly sinuous in the upper levels, and as they escape from the eastward drift they become more nearly cyclonic. In the component vectors there is a maximum velocity in the 3,000-meter level, and a decrease in velocity upward and downward. The cyclonic disturbances often penetrate to the 10,000-meter level, but in many cases they cease before the cirrus level is reached, and as a practical matter they do not retain an important value above 4,000–5,000 meters, because the eastward drift there predominates. It should be noted that in the cyclonic components the U-shaped opening at *A* is in the northwest quadrant of the upper levels, but in the northeast quadrant of the lower levels; in the anticyclonic components it is in the southwest quadrant in all levels. As can be readily inferred, it becomes a very difficult task for the meteorologist to construct analytical formulas to cover these complex curves, and to include the pressures, temperatures, and vectors under the general equations

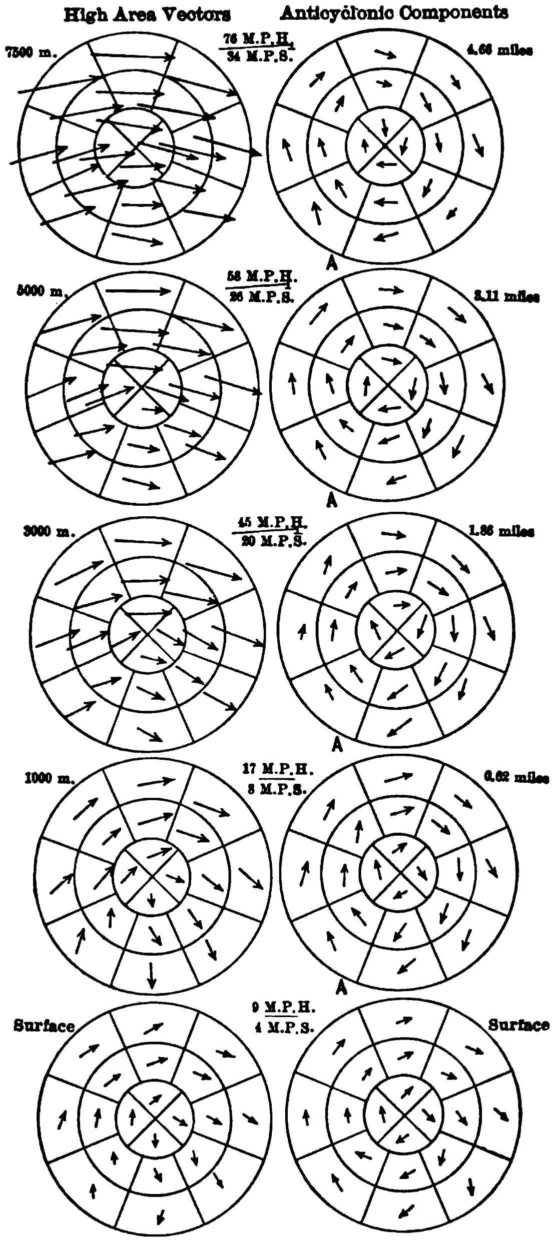


Fig. 45. Mean vectors of velocity and direction in high areas.
1 *cm.* = 500 kilometers for the distances.
1 *mm.* = 2 meters per second = 4.48 miles per hour for the velocity vectors.

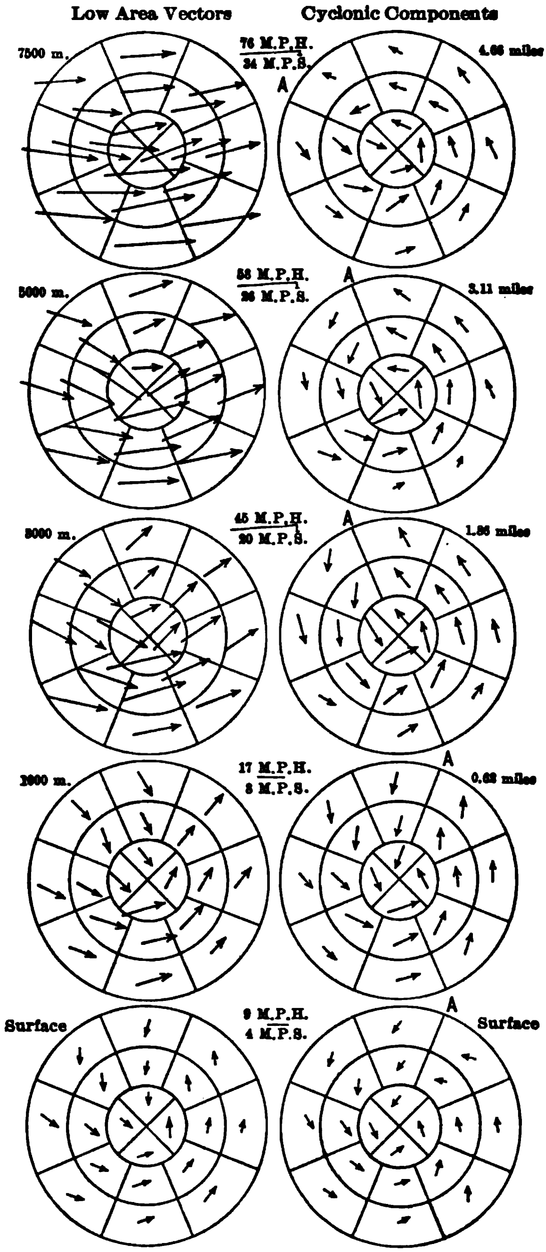


FIG. 46. Mean vectors of velocity and direction in low areas.

1 *cm.* = 500 kilometers for the distances.

1 *mm.* = 2 meters per second = 4.48 miles per hour for the velocity vectors.

of motion requires unusual skill. Some years may, therefore, elapse before a satisfactory general theory can be perfected. In what follows there is only possible a series of fragmentary propositions regarding circulations in the atmosphere.

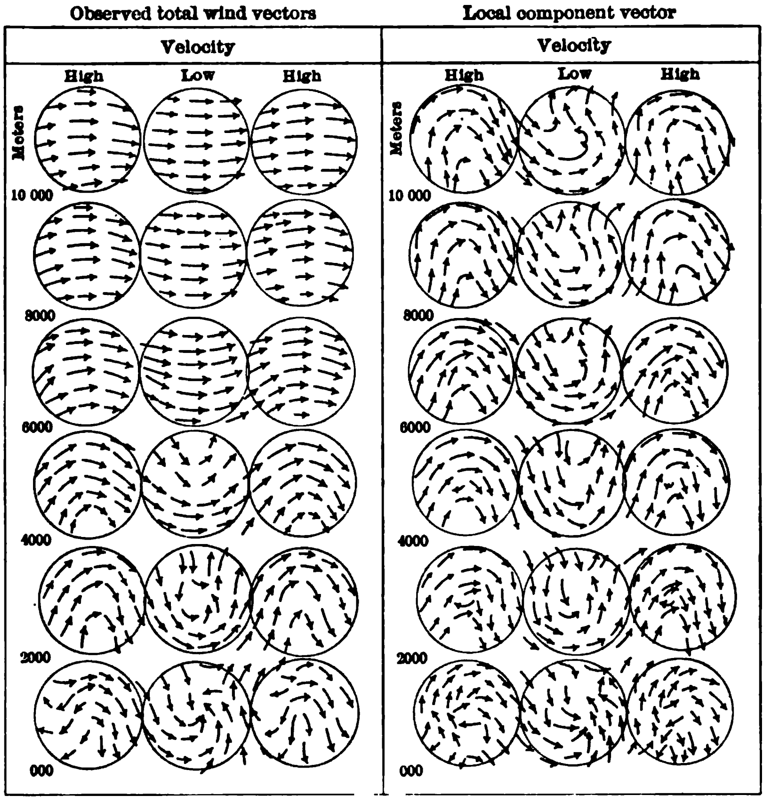


FIG. 47. Observed total and observed local component wind vectors connecting high and low areas.

The Land Cyclone

It has been shown that the ocean cyclones are imperfect vortices of the dumb-bell-shaped typé, which depart from the nearly perfect forms found in hurricanes and tornadoes, through the effect of certain resistances that are represented by component reversing vortices. These departures may become very irregular, leaving only the remnants of pure vortex motion from point to point in the cyclone, in proportion as the new system of thermodynamic forces, due to gravity acting on masses of air of different temperatures, are not symmetrically distributed about an axis. In the pure vortex motion of the tornado and the hurricane there was no need to consider specifically the action of gravity on the vortex motion, because of the symmetrical disposition of the air masses in superposed horizontal layers. In the cyclones, on the other hand, the differential action of gravity on adjacent air masses of different densities becomes the primary consideration, as demonstrated in Chapter II, so that the vortex action, though still of influence, becomes of secondary dynamic value. The study of temperature distributions in cyclones and anticyclones, together with the corresponding velocity vectors and pressures, must be first determined by observations before the dynamic theories can be suitably applied. The ocean cyclone has been used as a transition between the hurricane and the land cyclone, in order to bring out the method of the composition of vortices. In the land cyclone the departures from the perfect dumb-bell vortex are very great, especially in the upper strata, where the head of the vortex is depleted by its intrusion into the rapid eastward drift whose average velocities increase with the height above the surface of the ground. This subject is so very voluminous that only the leading features can be brought out in this place.

The cloud observations, made by the United States Weather Bureau, 1896–1897, showed that when the true cyclonic components of velocity are eliminated from that of the eastward drift, there remain a cold current on the western side of a cyclone and a warm current on the eastern side, and that this arrangement persists in a general way from the ground up to the cirrus levels, 10,000 meters. The tangential velocities v are at a maximum in the strato-cumulus level, 3,000 meters, and they decrease downward and upward, the lower part being the truncated portion of the vortex, while the longer upper part is gradually destroyed by degradation in the eastward drift. The radial velocities seem to be inward from top to bottom, taking the cyclone as a whole; or rather the inward flow on the west and the outward flow on the east side do not appear to balance in the different levels, so that the mean velocity shall become inward below and outward above, as required by the perfect dumb-bell vortex. In this respect the funnel-shaped vortex, with the tube pointing upward, was suggested as the proper mode of analysis, but the analogy does not hold in its details. The determination of these radial velocities, upon which so much depends, in the upper strata is really very difficult, and some suitably located observatory might properly devote several years of observations to the elucidation of this point with precision. It has been proper to make a résumé of the observations, so far as was required to bring out the theory of the subject. We began with the pressures and then took up the temperatures and the velocities in the levels up to 10,000 meters.

The land cyclone differs from the ocean cyclone especially in the fact that it is not so highly developed as a dumb-bell-shaped vortex. The barometric pressure in the ocean cyclone sometimes falls to 28.00 inches (711 mm.), while in the land cyclone it seldom falls below 29.00 inches (737 mm.). This deficiency of the central areas in the vortex tubes is due to a variety of causes, but the principal fact is that the air masses of different temperatures are placed side by side on the same horizontal plane instead of being superposed; and the second point is that the penetration of the head of the vortex into the eastward drift of the general circulation is followed by its depletion, which is caused by stripping away from the vortex of fragments of the masses of ascending air. The meteorological data that serve to illustrate these facts can be briefly presented.

A study has been made of the location of the isobars, the variations in the temperature, and the wind velocity and direction in a large typical land cyclone, by constructing the mean values for a composite of nine selected cyclones, March 16, 1876, March 27, 1880, April 18, 1880, January 12, 1890, December 3, 1891, November 17, 1892, April 20, 1893, January 25, 1895, November 22, 1898. They were chosen such that the cyclonic center occupied nearly the same place in the United States, namely, the lower Ohio Valley, and they were about equally developed at the axis. On the weather map the scale is 1 mm. = 10,000 meters. The linear distances of the radii to each isobar

TABLE 57

The Land Cyclone

	(1)	(2)	(3)	(4)	(5)	(6)
I. The Radial Dimensions						
B	760	755	750	745	740	735
ϖ	1250000	975000	733000	508000	300000	110000
$\log \varpi$	6.09691	5.98900	5.86510	5.70586	5.47712	5.04139
$\log \rho$		0.10791	0.12390	0.15924	0.22874	0.43573
II. Δt Variation of the Temperature from the Mean Distributions						
S	−1.3	−1.5	−1.1	−0.5	+0.2	+0.9
S 30 E	+1.3	+1.7	+1.9	+1.4	+2.3	+2.7
S 60 E	+2.9	+2.8	+2.9	+2.9	+3.0	+3.2
E	+2.1	+2.1	+2.8	+2.8	+2.4	+2.1
E 30 N	+1.7	+1.8	+1.7	+1.7	+1.2	+1.0
E 60 N	+2.2	+2.3	+2.1	+1.8	+1.6	+1.1
N	0.0	+0.1	+0.1	+0.7	+0.8	+1.3
N 30 W	−1.1	−1.2	−1.0	−0.4	+0.4	+1.3
N 60 W	−3.2	−3.0	−2.8	−2.4	−1.0	+0.8
W	−6.4	−5.5	−4.3	−2.7	−0.7	+1.7
W 30 S	−7.1	−6.1	−4.7	−3.3	−1.1	+1.0
W 60 S	−5.6	−4.6	−3.3	−2.1	−0.4	+1.1
III. Wind Velocity and Direction with the Isobar						
S	6.4 47°	7.3 44°	10.0 43°	11.9 43°	12.6 44°	12.6 46°
S 30 E	6.0 45	7.6 46	9.1 46	10.3 44	10.8 41	11.0 38
S 60 E	6.7 37	7.8 41	8.7 44	9.4 45	9.9 39	10.1 35
E	6.9 40	7.6 42	7.2 43	6.8 41	7.3 39	8.5 37
E 30 N	7.9 42	8.1 45	8.5 46	8.2 46	7.9 45	8.2 45
E 60 N	7.8 29	7.6 35	7.8 41	8.0 44	7.8 43	7.6 40
N	6.5 54	7.3 54	9.0 54	11.1 53	10.9 50	9.3 47
N 30 W	6.6 55	8.7 51	10.8 48	11.6 45	10.4 42	9.6 39
N 60 W	8.1 49	10.2 46	11.0 43	11.0 41	10.4 39	10.2 38
W	8.9 44	10.8 40	11.4 38	11.5 35	11.3 33	11.5 31
W 30 S	6.9 39	8.4 36	9.8 35	10.8 34	11.1 34	11.2 34
W 60 S	6.7 51	9.4 51	10.7 52	11.2 50	11.1 45	10.7 39

were scaled in the N.W. – S.E. and S.W. – N.E. directions, and the means were taken for the equivalent circular isobars. All the data of Table 57 are given in the metric measures. The section I contains the barometric pressure B, the circular radii

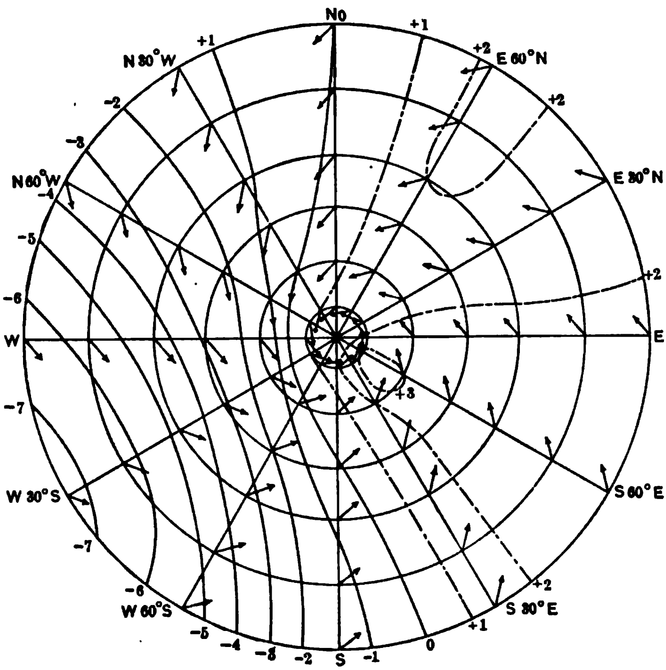


FIG. 48. Land cyclone with circular isobars equivalent to the elongated cyclones of the United States, with the temperature disturbances and the wind vectors, the center being located in the central valleys.

ϖ, and the $\log \rho = \log \frac{\varpi_n}{\varpi_{n+1}}$. In a pure vortex $\log \rho =$ constant, but in the land cyclone $\log \rho$ is not a constant. Hence it follows that the pure vortex laws no longer prevail, though their influence continues to be felt.

On Fig. 48 the isobars are laid down, and radii are drawn for every 30 degrees, making points of intersection where the computations can be concentrated. The isobars are spaced too widely near the center of the land cyclone, that is, the barometric pressure does not fall near the axis sufficiently to conform to the outer isobars from which the vortex is to be constructed. The temperatures were scaled from the weather maps at the 72 points of intersection just indicated, and the means for each point were taken and plotted on diagrams, one for each radius. As a matter of fact, the compilation was made in the English system, and this involved 132 readings for each of the nine cyclones. Similarly, the wind directions relative to the isobars, and the velocities were measured at the same points. The results transformed to metric measures appear in sections II, III, of Table 57. After the temperatures at the several points had been found, it was necessary to subtract from them the average undisturbed temperature of the region, that is, the normal temperature for the average of the dates of the years in question. It is desired to know what disturbance of temperature accompanies the cyclonic movements of the air, as distinguished from the normal temperatures which are due to the general circulation taken by itself. The section II contains these differences, which are also plotted on Fig. 48, together with the vectors of the wind circulation.

It shows that the maximum departure for the cold area is on the S.W. edge of the cyclone, and that the maximum departure for the warm area is in the S.E. quadrant generally, the line of 0-departure running nearly due north and south through the center of the cyclone. The mean angle of the vector is $i = -43°$, though it ranges from 27° to 54° in an irregular fashion from one point to another. It has been shown that a similar asymmetric distribution of the temperature prevails within the lower levels, being at a maximum of departure in the strato-cumulus level, 3,000 meters, and disappearing above in the cirrus level, 10,000 meters. When there are masses of air of different temperatures on the same level, the densities are different, and the action of gravity is to set up currents which cause the cold currents to underflow the warm currents, and the warm currents to overflow the cold currents. The effort of gravity is to restore the isobars to a normal value when they have been disturbed by abnormal temperature densities. The air masses are transported from the north or from the south into some middle latitude, where this underflowing and overflowing process sets up the cyclonic and the anticyclonic circulations. This principle can be illustrated by a vertical section running from west to east through a series of cold and warm masses of air. In a cold mass the isobars are concentrated near the surface and opened in the upper levels; in a warm mass the upper isobars are concentrated and the lower are opened. When these cold and warm masses alternate with one another the cold underflows in two opposite directions, and the warm overflows in two opposite direcions. In effect in nature, the cold mass from the

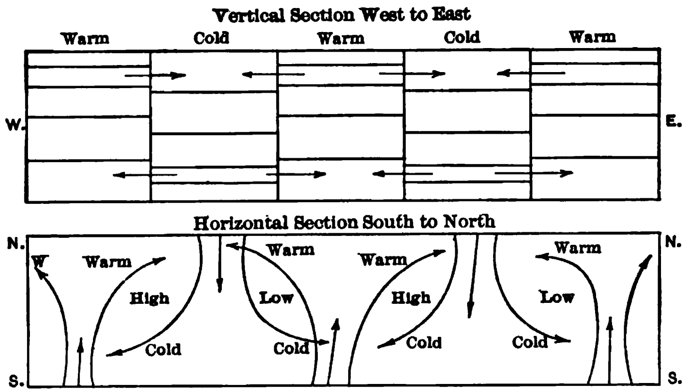


FIG. 49. Model of the action of gravity *G* in forming streams of air which underflow and overflow the warm and cold masses on either side. The isobars in warm masses are relatively open below and closed above; in cold masses they are relatively closed below and open above. Gravity tends to restore them to the same barometric levels, and the cyclones and anticyclones are the effect of this process of circulation in the impure vortices of the dumb-bell-shaped type. A thermodynamic discussion of the cyclone and the anticyclone has been given in Chapter II, showing the interplay of the general forces: gravity, pressure, circulation, and radiation. We shall next give a summary of the data for the land cyclone, which will include the entire series of terms in the equations of motion, transformed to the vortex type, as in Formulas (561), and their various modified forms.

north flows southward and divides, underflowing two warm masses on either side, while the warm mass flows northward and overflows two cold masses on either side. The result of this complex system of currents is to produce the cyclones and anticyclones, and the tendency is to approach a dumb-bell vortex, though the resistance is too great in general to permit this to be done.

Recapitulation of the Formulas for the Dumb-Bell-Shaped Vortex, (526)–(550), *Fig.* 18

We resume the formulas for the dumb-bell-shaped vortex in connection with the cylindrical equations of motion, (526) to (550) and (406), illustrated by Figs. 14, 18, including all the terms: inertia, expansion, deflection, friction, radiation, circulation, pressure, and gravitation. If $+ i$ is the angle between the tangent and the horizontal velocity of motion, positive (+) on the outside, negative (−) on the inside of the circle, we have $a z = 90° + i$, the angle from the radius, which in the complete vortex passes from $a z = 0°$ and $i = - 90°$ on the lower reference plane for the air inflowing along the radius, to $a z = 90°$ at the middle height and $i = 0°$ tangential, to $a z = 180°$ and $i = + 90°$ on the upper reference plane for air outflowing along the radius. The intermediate inflowing and outflowing angles are all determined by the relation of the line integral of the air flowing radially to or from the axis, and the surface integral of the air that rises in the vortex from the surface upwards. For $a z = 60°$, $i = - 30°$ and for $a z = 120°$, $i = + 30°$. Hence, we have for

(646) *Angles*,
$$- \cos a z = + \sin i,$$
$$+ \sin a z = + \cos i,$$
$$- \cot a z = + \tan i = \frac{u}{v}.$$

(647) *Velocities*,
$$u = - A a \varpi \cos a z = + A a \varpi \sin i,$$
$$v = A a \varpi \sin a z = + A a \varpi \cos i,$$
$$w = 2 A \sin a z = + 2 A \cos i,$$
$$\frac{u}{v} = - \cot a z = + \tan i,$$
$$\frac{u}{w} = - \frac{a \varpi}{2} \cot a z = \frac{a \varpi}{2} \tan i,$$
$$w = - \frac{2 u}{a \varpi} \tan a z = \frac{2 u}{a \varpi} \cot i.$$

(648) Line Integral, $2\pi \varpi u = - 2 \pi A a \varpi^2 \cos a z = 2 \pi A a \varpi^2 \sin i.$

(649) Surface Integral, $\pi \varpi^2 w = + 2 \pi A \varpi^2 \sin a z = 2 \pi A \varpi^2 \cos i.$

(650) Ratio, $\frac{2\,\pi\varpi u}{\pi\varpi^2\,w} = \frac{2\,u}{\varpi w} = -\,a \cot a\,z =$

$a \tan i = a\,\frac{u}{v}.$

(651) Tangential Angle, $\tan i = \frac{1}{a} \cdot \frac{2\,\pi\varpi\,u}{\pi\varpi^2\,w} = \frac{2\,u}{a\varpi\,w} = -$

$\cot a\,z = \frac{u}{v}.$

The Meaning of the Tangential Angle i

The line integral of velocity is the product of the closed line, circle, ellipse, or any other boundary line, multiplied by the mean velocity at right angles to it, or more properly the integral of the mass velocity per unit length around the boundary,

$$\sigma = \int u\,d\,s$$

where u is the velocity perpendicular to $d\,s$ at every point. For a circular vortex symmetrically disposed to an axis at the radius ϖ the velocity u is radial and the same at every point of the boundary circle $2\,\pi\,\varpi$, so that the line integral is,

$$S = 2\,\pi\,\varpi\,u$$

It is assumed that the inflowing air at the bottom of a vortex, for example in a tornado, is not congested and compressed, and therefore the inflowing mass must escape by rising upward from one level plane to another in the surface integral $\Sigma = \int\int w\, d\,S$.

The inflowing mass $2\,\pi\varpi\,u$ escapes vertically with the velocity w through the plane whose area is $\pi\varpi^2$, so that the surface integral is simply $\pi\varpi^2\,w$, not now counting the impermeable bottom, or the cylindrical surface. The same mass of air entering the vortex radially and horizontally on one plane escapes vertically on the next adjacent horizontal plane, and it is the vortex-constant a and the tangential angle i that controls this flow, through the equation,

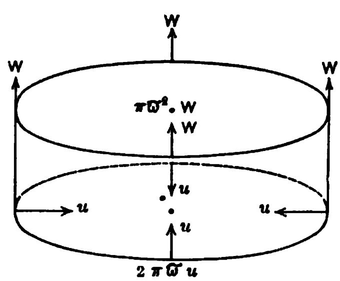


FIG. 50. The line integral and the surface integral in vortices.

$$\frac{u}{w} = \frac{a\varpi}{2}\tan i = \frac{v}{w}\tan i$$

Hence, $\tan i = \frac{1}{a}\cdot\frac{2\pi\varpi u}{\pi\varpi^2 w} = \frac{2u}{a\varpi w} = \frac{u}{v}$, and $a \tan i =$ the ratio of the line integral to the surface integral.

Similarly from (523) to (525) for the funnel-shaped vortex, and from (527) to (529) for the dumb-bell-shaped vortex, we may summarize,

	Funnel Vortex	*Dumb-bell Vortex*
(652)	$\frac{u}{v} = \frac{1}{z}.$	$\frac{u}{v} = -\cot a z = \tan i.$
	$\frac{u}{w} = -\frac{\varpi}{2}\cdot\frac{1}{z}$	$\frac{u}{w} = -\frac{\varpi}{2} a \cot a z = \frac{\varpi}{2} a \tan i.$
	$\frac{v}{w} = -\frac{\varpi}{2}.$	$\frac{v}{w} = \frac{\varpi}{2} a.$
	$w = -\frac{2}{\varpi}. z . u.$	$w = \frac{2}{\varpi}.\frac{1}{a \tan i}. u.$

The connecting link between these vortices becomes,

$$(653) \qquad -\frac{1}{a z} = \tan i,$$

the difference of sign depending upon the fact that the vertical axis was assumed in opposite directions in these vortices. The tangential angle i varies from one plane to another, $-90°$ on the lower reference plane, gradually changing to $0°$ on the middle plane where there is no inflowing or outflowing air, then continuing to $+90°$ on the upper reference plane. These are due entirely to the supply of air needed to balance the inflowing or outflowing line integrals with the increasing or decreasing vertical surface integrals over the same planes in succession.

It has been customary for meteorologists to explain these tangential angles in cyclones as the effect of the deflecting or the friction forces. Thus, by equations (480) (481),

$$(654) \qquad \tan i_1 = \frac{u}{v} = -\frac{k-c}{\lambda}, \text{ for the inner part,}$$

$$\tan i_2 = \frac{u}{v} = -\frac{k}{\lambda}, \text{ for the outer part,}$$

but in fact the theory is erroneous. The deflecting forces dependent upon λ are small, and those depending on the friction

k are nearly negligible. The inflowing tangential angle in cyclones at the surface is due to the supply of air necessary to compensate for the rising air over the entire surface of the closed isobar, and therefore depends upon the integral of the entire thermodynamic system. In the hurricane the inflowing angle on the ocean shows the amount of air that is required to supply the mass of air that is flowing away in horizontal radial directions in the high levels, underneath the cold stratum that has flowed as a sheet over the tropic region. Since $\tan i = -\cot a z$, the vertical distance of the azimuth plane from the reference planes can at once be found and thence the structure of the entire cyclonic vortex can be deduced by the preceding methods. Since we are not dealing with pure vortices in the case of cyclones, these simple laws must be modified from point to point according to conditions, and we proceed to evaluate the land cyclone in the several complete terms of the equations of motion, applied to the dumb-bell system. Making the substitutions indicated, we find the following system of equations for a symmetrical circular vortex with (406) in cylindrical co-ordinates.

(655) Radial, $$-\frac{\partial P}{\rho \partial \varpi} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \varpi} + w\frac{\partial u}{\partial z} - \frac{v^2}{\varpi} - 2\,\omega_3 \cos\theta \,.\, v + ku + dQ_{\varpi}.$$

$$-\frac{\partial P}{\rho \partial \varpi} = \frac{\partial u}{\partial t} + A^2 a^2 \varpi - 2\,\omega_3 \cos\theta \,.\, A\, a \varpi \cos i + k\,u + d\,Q_{\varpi}.$$

Tangential, $$0 = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial \varpi} + w\frac{\partial v}{\partial z} + \frac{u v}{\varpi} + 2\omega_3 \cos\theta \,.\, u + k\,v + d\,Q_{\phi}.$$

$$0 = \frac{\partial v}{\partial t} + \ldots + 2\,\omega_3 \cos\theta . A\, a\, \varpi \sin i + k v + d\,Q_{\phi}.$$

Vertical, $$-\frac{\partial P}{\rho \partial z} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial \varpi} + w\frac{\partial w}{\partial z} + g + k\,w + d\,Q_z.$$

$$-\frac{\partial P}{\rho \partial z} = \frac{\partial w}{\partial t} - 4\,A^2\, a \sin i \cos i + g + k\,w + d\,Q_z$$

These equations contain the inertia, convergence, centrifugal, deflecting, friction, and radiation terms in succession, reduced to a form for computation at any point where the data are known.

Example of the Evaluation of the Terms in the Equations of Motion for a Cyclone

As an example in the evaluation of the several terms in the radial, tangential, and vertical equations of motion for a cyclone, I have taken the data of Table 57 for $B. P. T. \rho. R. \varpi. q. - i$, as in Table 58A, and have computed a, $a\,\varpi$, A, u, v, w, $\Delta\,\varpi$, $\Delta\,P$, ρ_m, in succession. In Table 58B is a summary of the several terms, as indicated in the equations of motion, for the pressure, convergence, deflection, inertia, and friction combined with radiation. In the vertical component the data and results are from Table 25, $(Q_1 - Q_0)$ in the first column for the levels 000 to 500 meters, these data being divided by 500 to give the heat losses per meter in a vertical direction. The other data of Tables 58B are reduced to the unit length in all cases for comparisons. The pressure term radially is very much larger than the sum of the convergence, deflection, and inertia terms, so that the remainder friction plus radiation amounts on the average to .0014582 per meter in mechanical units (M. K. S.). Since pressure acts inward these two terms act outward along the positive radius. The result shows how impossible it is to balance the terms of the equation of motion without friction and radiation. It is not now known how to separate friction energy from radiation energy directly, and thus evaluate them separately. These terms in the tangential component are much smaller, and act in the antirotational direction. The radial component averages 0.00146, and the tangential component is 0.00038, that is about one-fourth the amount. It is probably true that most of the tangential term is due to friction alone, and since i = about 45° we may suppose that the same amount of friction energy applies to the radial component, leaving 0.00108 for the transported heat energy towards the axis of the cyclone. The vertical component is without friction and amounts to 0.11648, so that more than 100 times as much heat is transported upward vertically as inward radially.

TABLE 58A

SUMMARY OF THE DATA OF OBSERVATIONS FOR THE VORTEX TERMS IN A TYPICAL CYCLONE

Quantities	(1)	(2)	(3)	(4)	(5)	(6)	Formulas
Barom. pressure, B	0.7600	0.7550	0.7500	0.7450	0.7400	0.7350	m— mercury
Force pressure, P	101323	100656	99988	99323	98655	97990	(M.K.S.) system
Temperature, T	290.7	287.1	284.9	283.2	281.9	281.1	South
	281.8	281.8	283.1	283.0	282.6	281.7	East
	262.9	266.3	269.2	272.7	275.8	279.0	North
	266.5	268.1	270.1	273.1	276.6	281.2	West
Density, ρ	1.2143	1.2215	1.2227	1.2219	1.2193	1.2145	South
	1.2526	1.2444	1.2305	1.2227	1.2163	1.2119	East
	1.3427	1.3475	1.2941	1.2689	1.2462	1.2237	North
	1.3246	1.3080	1.2897	1.2613	1.2427	1.2140	West
Gas coefficient, R	287.033	287.033	287.033	287.033	287.033	287.033	Constant
Radii, $\bar{\varpi}$	1250000	975000	733000	508000	300000	110000	Meters
Vortex constant, a	0.015	=	$\frac{180°}{12000}$	=	$\frac{180°}{9000 + 3000}$		45° truncated
Vortex product, $a\bar{\varpi}$	18750	14625	10995	7620	4500	1650	
Tube constants, A	.000341	.000499	.000910	.001562	.002802	.007633	S $A = \frac{u \operatorname{cosec} i}{a\bar{\varpi}}$
	357	519	655	986	1621	5156	E
	347	500	829	1457	2422	5635	N $= \frac{v \sec i}{a\bar{\varpi}}$
	475	738	1037	1510	2513	6966	W
Velocities, q	6.4	7.3	10.0	11.9	12.6	12.6	S Meters per sec.
	6.9	7.6	7.2	6.8	7.3	8.5	E
	6.5	7.3	9.0	11.1	10.9	9.3	N
	8.9	10.8	11.4	11.5	11.3	11.5	W
Tangential angle, $-i$	47°	44°	43°	43°	44°	46°	S Inflowing angle
	40	42	43	41	39	37	E
	54	54	54	53	50	47	N
	44	40	38	35	33	31	W
Radial velocity, u	−4.68	−5.07	−6.82	−8.12	−8.76	−9.06	S $u = q \sin i$
	−4.30	−5.08	−4.91	−4.46	−4.59	−5.12	E
	−5.26	−5.91	−7.37	−8.87	−8.35	−6.80	N $= A a\bar{\varpi} \sin i$
	−6.19	−6.94	−7.02	−6.60	−6.16	−5.92	W
Tangential velocity, v	4.36	5.25	7.31	8.70	9.06	8.76	S $v = q \cos i$
	5.29	5.65	5.26	5.13	5.67	6.79	E
	3.82	4.29	5.35	6.68	7.09	6.34	N $= A a\bar{\varpi} \cos i$
	6.40	8.27	8.98	9.41	9.47	9.86	W
Vertical velocity, w	.000465	.000718	.001458	.002283	.004027	.010618	S $w = \frac{2v}{a\bar{\varpi}}$
	564	773	957	1346	2520	8230	E
	408	587	973	1753	3116	7685	N $= 2A \cos i$
	683	1181	1633	2470	4209	11952	W $= \frac{2u}{a\bar{\varpi}} \cot i$
Differences $\Delta\bar{\varpi}$	....	275000	242000	225000	208000	190000	The mean values of two successive tubes placed under the second of the pair from which they are computed.
Differences ΔP	....	667	668	665	668	665	
Means ρ_m	....	1.2179	1.2221	1.2223	1.2206	1.2169	
	....	1.2485	1.2375	1.2266	1.2195	1.2141	
	....	1.3451	1.3208	1.2815	1.2576	1.2350	
	....	1.3163	1.2989	1.2755	1.2520	1.2284	

The ratio of the line integral to the surface integral checks.

$$\frac{2\pi\bar{\varpi}u}{\pi\bar{\varpi}^2 w} = a \tan i = \frac{2u}{\bar{\varpi}w} = \frac{au}{v}.$$

TABLE 58B.

EVALUATION OF THE EQUATIONS OF MOTION

Radial. $-\frac{P_1-P_0}{\rho_{10}(\varpi_1-\varpi_0)}=A^2a^2\varpi-2\varpi_3\cos\theta\,.\,A\,a\,\varpi\cos i+\int\frac{\partial u}{\partial t}+ku+(Q_1-Q_0)$

Terms	(1)	(2)	(3)	(4)	(5)	(6)	Formulas
South							
Pressure	....	.0019915	.0022587	.0024180	.0026311	.0028762	$-\frac{(P_1-P_0)}{\rho_{10}(\varpi_1-\varpi_0)}$
Convergence	....	.0000437	.0000855	.0002078	.0004046	.0009861	$(A^2a^2\varpi)_m$ mean
Deflection	....	4507	5889	7510	8334	8353	$-(2\varpi_3\cos\theta\,.\,A\,a\,\varpi\cos i)_m$
Inertia	....	70	430	432	260	141	$\partial u/\partial t$
Sums	....	.0005014	.0007174	.0010020	.0012640	.0018355	
Friction + radiation	....	.0014901	.0015413	.0014160	.0013671	.0010407	.0013710 $ku+(Q_1-Q_0)\varpi$
East							
Pressure	....	.0019506	.0022358	.0024096	.0026335	.0028818	
Convergence	....	.0000475	.0000649	.0000913	.0001446	.0004176	
Deflection	....	5047	5113	5136	5324	5841	
Inertia	....	133	−35	−95	28	136	
Sums	....	.0005655	.0005727	.0005954	.0006798	.0010153	
Friction + radiation	....	.0013851	.0016631	.0018142	.0019537	.0018665	.0017365

North								
Pressure	....	.0018032	.0020899	.0023063	.0025538	.0028340		
Convergence	....	.0000443	.0000840	.0001780	.0003195	.0005910		
Deflection	....	3804	4523	5643	6417	6256		
Inertia	....	132	400	541	−215	−619		
Sums	....	.0004379	.0005763	.0007964	.0009397	.0011547		
Friction + radiation	....	.0013653	.0015136	.0015099	.0016141	.0016793	.0015364	
West								
Pressure	....	.0018426	.0021251	.0023172	.0025652	.0028492		
Convergence	....	.0000916	.0001485	.0002190	.0003435	.0008138		
Deflection	....	6881	8088	8630	8864	9064		
Inertia	....	180	23	−127	−135	−76		
Sums	....	.0007977	.0009596	.0010693	.0012164	.0017126		
Friction + radiation	....	.0010449	.0011655	.0012479	.0013488	.0011366	.0011887	.0014582

Tangential. $0 = 2\,\omega_3 \cos\theta .\, A\, a\, \varpi \sin i + \int \frac{\partial v}{\partial t} + k v + (Q_1 - Q_2)_\phi.$

	(1)	(2)	(3)	(4)	(5)	(6)		
South								
Deflection	....	.0004277	.0004129	.0004096	.0004134	.0004242		
Inertia	....	− 158	− 506	− 462	− 146	+ 141		
Friction + radiation	....	−.0004119	−.0003623	−.0003634	−.0003988	−.0004383	−.0003949	
East								
Deflection	....	.0003935	.0004053	.0004014	.0003856	.0003693		
Inertia	....	− 62	+ 96	+ 43	− 219	− 525		
Friction + radiation	....	−.0003873	−.0004149	−.0004057	−.0003637	−.0003168	−.0003777	
North								
Deflection	....	.0004853	.0004853	.0004822	.0004693	.0004490		
Inertia	....	− 84	− 261	− 441	− 166	+ 350		
Friction + radiation	....	−.0004769	−.0004592	−.0004381	−.0004527	−.0004840	−.0004622	
West								
Deflection	....	.0004011	.0003774	.0003567	.0003354	.0003179		
Inertia	....	− 332	− 174	− 143	− 24	− 183		
Friction + radiation	....	−.0003679	−.0003600	−.0003424	−.0003330	−.0002996	−.0003406	−.0003939

Vertical. $-\frac{P_1 - P_0}{\rho_{10}(z_1 - z_0)} = -4A^2 a \sin i \cos i + g + k w + (Q_1 - Q_0)_z$

Term	(1)	(2)	(3)	(4)	(5)	(6)	Means	
South								
Radiation	−.0918000	−.1080000	−.1224000	−.1300000	−.1384000	−.1410000	−.1219300	$(Q_1 - Q_0)_z$
East								
Radiation	−.1090000	−.1100000	−.1114000	−.1108000	−.1100000	−.1410000	−.1153700	
North								
Radiation	−.1002000	−.0900000	−.0812000	−.0920000	−.0972000	−.1410000	−.1002700	
West								
Radiation	−.1194000	−.1220000	−.1250000	−.1292000	−.1334000	−.1410000	−.1283300	−.1164800

NOTES.—The radiation throughout these summaries is expressed in mechanical units per meter length. The distance between isobars on the radius is about 200,000 meters. It would be possible to evaluate the transportation of heat in any direction, provided the friction terms $k u$, $k v$ could be eliminated. This subject will require a special research. These examples were computed for $\theta = 50°$, $\phi = 40°$. The relative transportation of heat is somewhat in the following ratios:

Vertical (z) upwards, −.1165
Radial (ϖ) inwards, +.0015
Tangential (φ) positive, −.0004

Similar studies of the relation of the heat and radiation terms to the mechanical dynamic terms may be extended to the higher levels of the cyclone, and to the anticyclone, but the final conclusion to be remembered is that there is no possibility of balancing in a dynamic system of equations the several terms of motion, without including the radiation of heat energy, and its convection from point to point. This branch of meteorology will require much further study along the lines that have been developed in this Treatise.

CHAPTER V

Radiation, Ionization, and Magnetic Vectors in the Earth's Atmosphere

The incoming solar radiation separates into two parts, the first the irreversible heat that cannot be transformed back into the original energy, the second the reversible energy which appears as electrical and magnetic forces. The heat energy is observed as the air temperature at different points, and its effects are found in the general and the local circulations of the atmosphere. The short waves of the solar radiation, at very high temperatures, as 6700° to 7700°, are capable of producing ions of positive and negative electricity by the disintegration of the atoms and molecules of the gases that compose the air, whereby a part of the radiation energy reappears by transformation as free ions, or free electric charges, more abundantly in some strata than in others. These ions tend to move in electric streams, in certain general lines as controlled by a series of physical conditions, and in their movement they induce magnetic deflecting vectors, which disturb the earth's normal magnetic field, through whose lines of magnetic force the electric ions move. We have, therefore, to study the distribution of the solar radiation in the atmosphere, the production of free electric charges through ionization, and the dependent induced magnetic deflecting vectors. In spite of prolonged researches in these subjects by many students, there is a wide discrepancy in the results, as whether the solar intensity of radiation is 2.00 calories or 4.00 calories, whether the absolute coefficient of electric conduction is 2×10^{-5} or 6×10^{-5}, whether the vectors that produce the diurnal variations of the magnetic field originate in the higher or the lower strata of the atmosphere, whether the sun is a star with a variable periodic output of radiation or practically constant, and whether the annual changes of the weather conditions are dependent upon solar variability or are merely accidental.

The Determination of the Intensity of the Solar Radiation by Observations with the Pyrheliometer and the Bolometer

Measurements of the intensity of the solar radiation are made by the pyrheliometer, which integrates in bulk the rays received by the instrument, and by the bolometer which measures the individual lines in the energy spectrum. The two instruments supplement each other, because while the pyrheliometer gives no account of the selective absorption of lines and bands in the spectrum, the bolometer defines these depletions, and permits the comparison of the observed spectrum energy with that of a full radiator at the given temperature. Neither instrument gives any account of that portion of the incoming solar radiation which is reflected back to space, as the albedo of the earth, but this can be found indirectly by thermodynamic computations on the temperatures of the atmosphere, as observed in balloon ascensions to great elevations.

The Pyrheliometer. This instrument consists of a chamber for receiving a bundle of the solar rays, whose temperature can be accurately measured at any time. The temperatures are measured when the solar rays are shaded by a screen, and again when exposed to the radiation, the sum of the changes of temperatures in a given interval of time, as one minute, being the effect of the radiation in temperature degrees. A factor can be found by experiment which will convert these temperature changes per minute into calories per square centimeter per minute. There are many types of actinometers or pyrheliometers, in which the different materials used for receiving the radiation are involved with certain conversion coefficients. The earliest form of pyrheliometer, by Pouillet, 1838, consisted of a silver vessel filled with a known volume of water, the surface being blackened to absorb all the radiation, whose increase in temperature in a given time could be measured by a thermometer whose bulb was embedded in it. Silver box with mercury, copper box with mercury, silver disk with no liquid, and many other combinations have been employed. The electrical resistance thermometer, the Ångström double strip compensated pyrheliometer, in which the heat absorbed is measured by a compensating electrical resistance, are used with success. We shall confine our attention to Abbot's silver disk pyrheliometer, which has been standardized against an elaborate Primary Standard No. III, 1911, of the Astrophysical Observatory of the Smithsonian Institution, and furnishes the comparison factors for copies of the secondary pyrheliometers. This instrument leaves little to be desired for durability and accuracy of its operation. More time is required for the observations with a thermometer system than with an electrical resistance apparatus, but the latter needs much more elaborate auxiliaries, battery, galvanometer, current, and resistance apparatus, so that it is less readily portable, and more liable to accidental inaccuracies of adjustment.

Theory of the pyrheliometer.

Let S = the entire surface of the body receiving radiation.
s = the cross-section of the rays falling upon it.
c = the coefficient of heat received referred to water, $dQ = c\,dT$.
h = the coefficient of heat lost by radiation, $dQ_2 = hSTdt$.
q = the intensity of the radiation received in dt, $dQ_1 = qsdt$.

Hence, the general equation of equilibrium is,

(656) $dQ = dQ_1 - dQ_2 = q\,s\,dt - h\,S\,T\,dt = c\,dT.$

The *shaded or cooling term.* If the body is in the shade, $q=0$, and we have,

(657) $\frac{dT}{T} = -\frac{hS}{c}dt.$ Integrate for $T=T_0$, when $t=0$.

(658) $\log T = -\frac{hS}{c}t + \text{const.} = -\frac{hS}{c}t + \log T_0.$

(659) $\frac{T}{T_0} = e^{-\frac{hS}{c}t}$ or $T = T_0\,e^{-\frac{hS}{c}t}.$

This is the cooling correction for T= the excess of the temperature above the surrounding medium, when t is the interval of time elapsed, and T_0 is the initial excess at which the cooling begins.

The exposed or heating term. Divide the general equation by c, and we have,

(670) $dT = \frac{qs}{c} dt - \frac{hS}{c} T dt.$

From (656) the maximum temperature T_m is obtained when $dT = 0$,

(671) $0 = qs - hST_m, \quad \frac{qs}{c} = \frac{hS}{c} T_m.$ Substituting in (670),

(672) $dT = \frac{hS}{c} T_m dt - \frac{hS}{c} T dt, \quad \frac{dT}{T_m - T} = + \frac{hS}{c} dt.$

Integrate for $T = T_0$ when $dt = 0$, since $dT = -d(T_m - T)$,

(673) $\log(T_m - T) = -\frac{hS}{c} t + \text{const.} = -\frac{hS}{c} t + \log(T_m - T_0).$

(674) $\log \frac{T_m - T}{T_m - T_0} = -\frac{hS}{c} t$, and $\frac{T_m - T}{T_m - T_0} = e^{-\frac{hS}{c} t}$. Hence,

(675) $T = T_0 e^{-\frac{hS}{c} t} + T_m \left(1 - e^{-\frac{hS}{c} t}\right)$ for the *total heating* effect.

This is the equation for the total effect of radiation when exposed to the solar rays. It consists of two parts: (1) the cooling term from the initial temperature T_0, and the heating term reckoned from the possible maximum temperature T_m relative to that of the surrounding medium, where T is the effective difference of temperature at the time t above the surrounding medium. It depends upon the total surface S of the receiving body, the cooling coefficient h, and the heating coefficient of the body c, referred to water.

An Example of the Practical Observations with the Silver Disk Pyrheliometer, S. I. No. 7, 1911

Several auxiliary tables must be prepared.

1. The table for the equation of time for every day of the year.
2. The table of the declination of the sun for every day of the year.
3. The table of 2 log r, for the radius vector of the earth.
4. The secant of the zenith distance, sec z, should be computed through the formulas leading to z.

TABLE 59. *La Quiaca*, Argentina. . . . Height 3462 meters. Latitude—22° 10′ Pyrheliometer S. I. No. 7 (0.610). . . .

1912	h m	s	I	Δ I	Means	Sum	1912	H m	s	I	Δ I	Means	Sum	
			°C							°C				
Sept. 22.	6.42	0	8.44	−.01			Sept. 22.	10.33	0	22.58	−.22			
		30	8.43	.00	.00				30	22.36	−.18	−.19		Shaded
	43	0	8.43	+.01				34	0	22.18	−.16			
		30	8.44						30	22.02				
	44	0	8.78	+.78				35	0	22.68	+1.10			
		30	9.56	.79	+.79				30	23.78	1.02	+1.06		Exposed
	45	0	10.35	.81	−.06	0.85		36	0	24.80	1.05	−.26	1.32	
		30	11.16						30	25.85				
	46	0	11.25	−.11				37	0	25.88	−.35			
		30	11.14	.12	−.12				30	25.48	.31	−.32		Shaded
	47	0	11.02	.14				38	0	25.17	.29			
		30	10.88						30	24.88				
	48	0	11.48	+.68				39	0	25.58	+.97			
		30	12.16	.72	+.68				30	26.55	.97	+.96		
	49	0	12.88	.64	−.18	0.86		40	0	27.52	.98	−.35	1.31	Exposed
		30	13.52						30	28.45				
	50	0	13.58	−.24				41	0	28.28	−.40			
		30	13.34	.22	−.23				30	27.88	−.40	−.38		Shaded
	51	0	13.12	.22				42	0	27.46	−.34			
		30	12.90						30	27.14				
						1.71							2.63	
	h.	m.	s.			.610		h.	m.	s.			.610	
	6	47	0					10	38	0				
	+	7	8		I	1.048			+7	8		I	1.604	Calories
					Log I	0.018						log I	0.205	
	6	54	8		sec s	4.65		10	45	8		sec s	1.15	
t	5	5	52				t	1	14	52				
δ	+0°	27′					δ	+0°	27′					
2 log r	0.	004					2 log r	0.	004					

(676) $\tan M = \frac{\tan \delta}{\cos t}, \tan A = \frac{\tan t \cos M}{\sin (\phi - M)}, \tan z = \frac{\tan (\phi - M)}{\cos A}.$

A carefully constructed diagram of curves is required for the interpolations to two decimals of sec z.

Take four readings of the thermometer every 30 seconds for the beam shaded and exposed alternately during ten minutes. Compute the mean ΔI for each group; take the means by pairs of the shaded terms, and subtract from mean exposed term lying between them; the sum of the two corrected ΔI gives the rise in the thermometer degrees per minute; multiply by the pyrheliometer factor 0.610 to obtain I in calories; take log I for the ordinate on the diagram of the Bouguer formula. Adopt as epoch the middle minute of the watch, running on mean time, and add the equation of time $+7^m\ 8^s$ to reduce to apparent time; subtract this from 12.00.00 for the hour angle t; take the sun's declination of date $\delta = +0° 27'$, and with t. δ as arguments read off sec z on the scale diagram for La Quiaca. Plot the points (log I, sec z) on the diagram, and draw the best mean line through the points of the day. Scale off the point (log I_0, sec $z = 0$), and reduce to the mean solar distance by the radius vector factor 2 log r to obtain log $I_{0 \cdot r}$ and $I_{0 \cdot r}$.

TABLE 60

LA QUIACA, ARGENTINA, SEPTEMBER 22, 1912

Similar pairs for September 22, 1912, at La Quiaca are as follows:

Time	Log I	Sec z	Time	Log I	Sec z	
A.M.			P.M.			
6.48.00	0.018	4.65	2.0.00	0.205	1.24	Plotted as ○ for the A.M.
6.58	0.060	3.89	2.10.0	0.182	1.33	Plotted as × for the P.M.
7.8	0 076	3.37	2.20.0	0.190	1.38	
8.21	0.157	1.82	4.0 .0	0.141	2.30	
8.31	0.180	1.73	4.10.0	0.124	2.52	
8.41	0.180	1.64	4.20.0	0.114	2.73	
10.28	0.202	1.17				
10.38	0.205	1.15				
10.48	0.202	1.13				

The most successful hours of observation are, weather permitting,

1. Group of five, ten minutes each, 7.00 to 7.50 A.M.
2. Group of three, ten minutes each, 8.30 to 9.00 A.M.

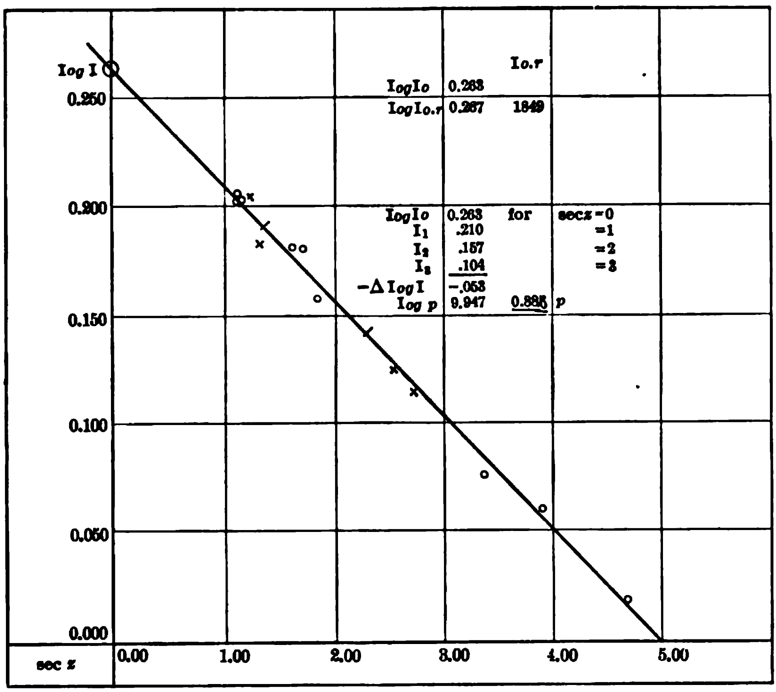


FIG. 51. Plotting the pyrheliometer observations.

3. Group of three, ten minutes each, 11.00 to 11.30 A.M.

The corresponding computations can be completed in one hour, so that three hours suffice to obtain $I_0.r$ on any good day.

The Bouguer Formula of Depletion

The incoming solar radiation is subjected to two types of depletion, (1) the reflection and scattering of the rays, on the molecules, ice crystals, dust, and other constituents of the atmosphere, whereby a certain amount of energy is reflected back to space as albedo, and is not subject to measurement by the pyrheliometer. This type of depletion affects chiefly the very short waves in the energy spectrum, and its region of operation is especially in the high cirrus region; (2) the other type of depletion is due to selective absorption of certain lines and bands of the spectrum, which can be determined specifically by bolometer observations, with the sun at different zenith distances for the same station, or by observations on the radiation at the same zenith distance, from stations having different heights above the sea level. The law of depletion is generally expressed by Bouguer's formula,

(677) $I = I_0 p^{\sec z}$,

where I_0 = the source of radiation before depletion,

p = the fraction which is transmitted for the unit distance,

$\sec z$ = the secant of the angular distance from the zenith to the incoming ray.

I = the radiation measured at the instrument.

The formula is only correct for homogeneous rays.

I. Passing to logarithms.

(678) $\log I = \log I_0 + \sec z . \log p$.

Taking two successive observations at z_1, z_2,

(679) $\log I_1 - \log I_2 = (\sec z_1 - \sec z_2) \log p$, hence,

(680) $\log p = \dfrac{\log I_1 - \log I_2}{\sec z_1 - \sec z_2}$, for p constant in the interval.

II. If it is assumed that $\sec z$ is constant and p variable,

First line, $\log I = \log I_0 + \log p . \sec z$.

Second line, $\log I' = \log I_0' + \log p' . \sec z$.

(681) $(\log I - \log I') = (\log I_0 - \log I_0') + (\log p - \log p') \sec z$.

(682) $\log \dfrac{I}{I'} = \log \dfrac{I_0}{I_0'} + \log \dfrac{p}{p'} \sec z$.

(683) $\dfrac{I}{I'} = \dfrac{I_0}{I_0'} \left(\dfrac{p}{p'}\right)^{\sec z}$.

By the first type of formula, it is seen that $\log I$ decreases as $\sec z$ increases. Taking $(\log I_1 - \log I_2)$ for unit differences in $(\sec z_1 - \sec z_2)$, as in the example, we easily find its mean value

− 0.053 for the given line as drawn, by reading the successive values of $\log I$ as the line crosses the sec z lines, 0.1.2.3 . . . whence p is found from $\log p$.

As the aqueous vapor rises from the surface upwards in the diurnal convection the absorption $(1.00 - p)$ increases, and p decreases towards the midday, so that the line joining the observed values of $\log I$ is likely to be a curve which is slightly convex toward the origin. In the dry climate at La Quiaca such a curvature was not observed, though it is common at such stations as Washington and Mt. Wilson. A refined treatment of the variable p can be made for the same zenith distance by the second form of the formula.

The Bouguer formula indicates that the depletion of the incoming radiation is proportional geometrically to the length of the path m which it traverses in the atmosphere, and it is common to the large class of physical formulas which correspond with similar conditions. It has been generally assumed, in applying the formula to the earth's atmosphere, that the unit distance is the depth of the atmosphere in the zenith m_0, and that what is measured by the pyrheliometer is the intensity of the solar radiation at the actual distance of the earth, before any depletion takes place. It is proposed to give several arguments which show that this is a clear assumption, which does not correspond with the facts of observation.

1. The Bouguer formula contains two unknown terms, since sec z is a ratio of two distances,

$$(684) \quad \sec z = \frac{m}{m_0} = \frac{m'}{m_0'} = \frac{m''}{m_0''} = \ldots .$$

in which the denominator m_0, m_0', m_0'' is undetermined; and since only $\log I$ and $\log p$ can be deduced from the observations, $\log I_0$ is also to be considered an unknown term. For example, m_0 may refer to one plane above the observer as the cumulus stratum, m_0' may refer to another plane as the cirrus stratum, and m_0'' may refer to any other plane as the outermost layer of the earth's atmosphere. The two unknown terms ($\log I_0$, m_0) refer to the source of the radiation I_0 on that plane whose unit distance above the surface m_0 may be any plane in the atmosphere satisfying these conditions. If the incoming radiation is partly reflected back to space as albedo on the cirrus levels m_0', then

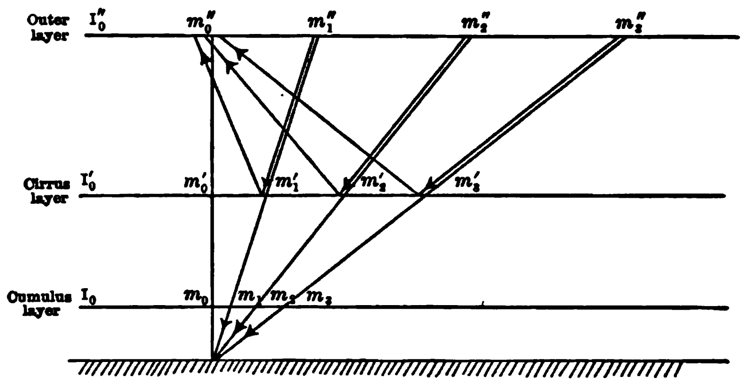


FIG. 52. Illustrating the use of the Bouguer formula.

I_0' is already diminished to that extent, and is not subject to observations by the instruments.

$$(685)\quad \sec z_1 = \frac{m_1}{m_0} = \frac{m_1'}{m_0'} = \frac{m_1''}{m_0''} = \dots .$$

$$\sec z_2 = \frac{m_2}{m_0} = \frac{m_2'}{m_0'} = \frac{m_2''}{m_0''} = \dots .$$

$$\sec z_3 = \frac{m_3}{m_0} = \frac{m_3'}{m_0'} = \frac{m_3''}{m_0''} = \dots .$$

The Bouguer formula contains, therefore, a double ratio,

$$(686)\quad \log\frac{I}{I_0} = \frac{m}{m_0}\log p,\ \text{that is}\ \frac{I}{I_0}\ \text{and}\ \frac{m}{m_0},$$

and it is quite indeterminate in itself, unless some means can be found to fix the unit distance m_0, whether I_0 emanates from the cumulus, the cirrus, or the outermost layer of the atmosphere. It has been customary to make $m_0 = 1$, refer it to the outermost layer, and thus assume that the pyrheliometer measures the so-called "solar-constant" I_0'', or intensity of the sun's radiation as it falls upon the earth's atmosphere, at m_0''.

TABLE 61

Examples of the Relative Steadiness of the Radiation at Stations of Different Elevations

I. Observations reduced to Abbot's Primary Standard, No. III, 1911

No.	La Quiaca 3462 −22°			Mt. Wilson 1780 +34°			Mt. Weather 526 +39°			Córdoba 438 −31°			Washington 34 +39°		
	Date 1912	$I_{0.r}$	p	Date 1906	$I_{0.r}$	p	Date 1908	$I_{0.r}$	p	Date 1912	$I_{0.r}$	p	Date 1909	$I_{0.r}$	p
1	Sep. 20	1.845	0.881	May 16	1.824	0.863	Jan. 2	1.867	0.845	Feb. 15	1.732	0.826	Sep. 25	1.718	0.869
2	21	1.849	0.853	June 9	1.742	0.897	5	1.774	0.851	16	1.765	0.818	28	1.698	0.873
3	22	1.849	0.885	19	1.746	0.903	6	1.738	0.857	17	1.760	0.825	29	1.770	0.794
4	23	1.854	0.883	July 3	1.750	0.877	9	1.722	0.832	21	1.791	0.877	30	1.694	0.845
5	25	1.854	0.859	17	1.675	0.895	14	1.755	0.851	27	1.640	0.852	Oct. 1	1.660	0.826
6	28	1.849	0.904	28	1.799	0.883	15	1.766	0.853	Mch. 4	1.622	0.852	6	1.783	0.716
7	29	1.828	0.887	Sep. 9	1.758	0.908	17	1.678	0.863	16	1.623	0.871	7	1.599	0.795
8	30	1.849	0.927	20	1.714	0.900	19	1.703	0.855	18	1.689	0.849	13	1.644	0.855
9	Oct. 1	1.849	0.918	Oct. 11	1.714	0.908	Feb. 6	1.787	0.815	19	1.727	0.813	19	1.678	0.871
10	2	1.854	0.834	18	1.702	0.920	8	1.791	0.833	23	1.730	0.795	22	1.791	0.824
Means	....	1.848	0.883	....	1.742	0.895	....	1.758	0.846	....	1.708	0.838	....	1.704	0.827

II. Variations on the Means

No.															
1	....	−.003	−.002	....	+.082	−.032	....	+.109	−.001	....	+.024	−.012	....	+.014	+.042
2	....	+.001	−.030	....	.000	+.002	....	+.016	+.005	....	+.057	−.020	....	−.006	+.046
3	....	+.001	+.002	....	+.004	+.008	....	−.020	+.011	....	+.052	−.013	....	+.066	−.033
4	....	+.006	.000	....	+.008	−.018	....	−.036	−.014	....	+.083	+.039	....	−.010	+.018
5	....	+.006	−.024	....	−.067	.000	....	−.003	+.005	....	−.068	+.014	....	−.044	−.001
6	....	+.001	+.021	....	+.057	−.012	....	+.008	+.007	....	−.086	+.014	....	+.079	−.111
7	....	−.020	+.004	....	+.016	+.013	....	−.080	+.017	....	−.085	+.033	....	−.105	−.032
8	....	+.001	+.044	....	−.028	+.005	....	−.055	+.009	....	−.019	+.011	....	−.060	+.028
9	....	+.001	+.035	....	−.028	+.013	....	+.029	−.031	....	+.019	−.025	....	−.026	+.044
10	....	+.006	−.049	....	−.040	+.025	....	+.033	−.013	....	+.022	−.043	....	+.087	−.003
Means	....	±.004	±.021	....	±.033	±.013	....	±.039	±.011	....	±.052	±.022	....	±.050	±.036

TABLE 62

ANNUAL SUMMARY OF THE MEAN VALUES OF THE PYRHELIOMETER OBSERVATIONS, REDUCED TO ABBOT'S PRIMARY STANDARD No. III, 1911

I. The mean annual radiation at the earth's mean distance $I_0 . r$

Station	Observer	ϕ	z	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912	Means
Mt. Whitney	Abbot	+37°	4420	......	......	......	......	......	......	1.863	1.860	......	......	1.861
La Quiaca	Bigelow	−22	3462	......	......	......	......	......	......	......	......	......	1.848	1.848
Humahuaca	Bigelow	−22	2939	......	......	......	......	......	......	......	......	......	1.798	1.798
Maimará	Bigelow	−23	2384	......	......	......	......	......	......	......	......	......	1.773	1.773
Mt. Wilson	Abbot	+34	1780	......	......	1.760	1.756	......	1.763	1.758	1.740	1.751	......	1.755
Jujuy*	Bigelow	−24	1302	......	......	......	......	......	......	......	......	......	1.768	1.768
Mt. Weather	Kimball	+39	526	......	......	......	......	......	1.732	......	......	......	......	1.732
Cordoba	Bigelow	−31	438	......	......	......	......	......	......	......	......	......	1.714	1.714
Washington	Abbot	+39	34	1.686	1.690	1.663	1.745	1.726	......	......	......	......	......	1.712
Washington	Kimball	+39	34	......	......	1.701	1.703	1.694	1.732	1.714	......	......	......	1.709

II. $I_0 . r$ reduced to the sea level by the bolometer factors

Station	Observer	F_1	z	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912	Means
Mt. Whitney	....	0.917	4420	......	......	......	......	......	......	1.709	1.706	......	......	1.707
La Quiaca	....	0.924	3462	......	......	......	......	......	......	......	......	......	1.708	1.708
Humahuaca	....	0.950	2939	......	......	......	......	......	......	......	......	......	1.708	1.708
Maimará	....	0.963	2384	......	......	......	......	......	......	......	......	......	1.707	1.707
Mt. Wilson	....	0.974	1780	......	......	1.715	1.711	[1.700]	1.717	1.712	1.695	1.705	......	1.706
Jujuy*	....	0.980	1302	......	......	......	......	......	......	......	......	......	1.733	1.733
Mt. Weather	....	0.993	526	......	......	......	......	......	1.720	......	......	......	......	1.720
Cordoba	....	0.997	438	......	......	......	......	......	......	......	......	......	1.709	1.709
Washington	....	1.000	34	1.686	1.690	1.663	1.745	1.726	......	......	......	......	......	1.702
Washington	....	1.000	34	......	......	1.701	1.703	1.694	1.732	1.720	1.704	......	......	1.709

* Used Abbot's recent data.

III. $I_0 . r$ reduced to the 10,000-meter level

	F_1												
Mt. Whitney	1.030	4420							1.919	1.916			1.917
La Quiaca	1.038	3462										1.918	1.918
Humahuaca	1.067	2939										1.918	1.918
Maimará	1.081	2384										1.917	1.917
Mt. Wilson	1.094	1780			1.926	1.921	[1.909]	1.928	1.923	1.904	1.915		1.918
Jujuy*	1.101	1302										1.946	1.946
Mt. Weather	1.115	526						1.932					1.932
Cordoba	1.120	438										1.919	1.919
Washington	1.123	34	1.893	1.898	1.868	1.956	1.938						1.911
Washington	1.123	34			1.910	1.912	1.902	1.945	1.932	1.914			1.919
Means on the sea level			1.686	1.690	1.693	1.720	1.707	1.723	1.714	1.702	1.705	1.713	1.711
Means on the cirrus level			1.893	1.898	1.901	1.930	1.916	1.935	1.925	1.911	1.915	1.918	1.918

* A longer series at Jujuy would probably give 1.743 at 1,302 meters, 1,708 at the sea level, and 1.918 at the 10,000-meter level

IV. The coefficient of transmission p

Mt. Whitney		4420											0.917
La Quiaca		3492										0.883	0.883
Humahuaca		2939										0.888	0.888
Maimará		2384										0.887	0.887
Mt. Wilson		1780			0.906	0.892							0.899
Jujuy		1302										0.791	0.791
Mt. Weather		526						0.832					0.832
Cordoba		438										0.842	0.842
Washington		34	0.712	0.865	0.815	0.813	0.799						0.821
Washington		34			0.798	0.818	0.819	0.805	0.829	0.830			0.816

The coefficient of transmission p falls into two mean groups $p_1 = 0.895$ above the 1,500-meter level, $p_0 = 0.820$ below that level

We shall now give strong evidence for thinking that the effective radiation I_0' is on the lower levels, its value being less than 2.00 calories, while about 2.10 calories is reflected back to space and escapes direct observation, so that the true "solar-constant" is really more than 4.00 calories per square centimeter per minute.

2. *The depletion of the incoming radiation from a maximum value on the cirrus levels, as determined by observations at different heights.*

Several series of pyrheliometer observations have been made at stations having different heights above the sea-level, and after they have been reduced to homogeneous data, and referred to a given primary standard, they are comparable, and show an increase in the value of $I_{0 \cdot r}$ with the height. The primary standard adopted is Abbot's Standard No. III, 1911, Smithsonian Astrophysical Observatory. The data of Table 13, Vol. II, Annals, have been reduced by the factor 0.951 and the data of the United States Weather Bureau (Mt. Weather Bulletin) by the factor 1.039, those observations having been there reduced previously to the Ångström pyrheliometer No. 104. In order to exhibit the average conditions that prevail at different elevations, several 10-day series have been impartially selected from the large amount of material in hand. The individual values ($I_{0 \cdot r}$, p) are given in Table 61, and in the second section their variations on the 10-day means. It is noted that the variations of the coefficient of transmission p, do not materially differ with the height, but there is a change in the mean values of p, approximately 0.825 for all stations below the 1,500-meter level, and 0.900 for all stations above that level. The variations of $\Delta I_{0 \cdot r}$ decrease steadily with the elevation, and $I_{0 \cdot r}$ increases with the height to a maximum value 1.918 which is to be identified as belonging approximately to the cirrus level. This is explained more fully on Table 62, where the available annual values are collected for eight stations. Special attention is directed to the remarkable results at La Quiaca.

It contains in section I, the station, observer, latitude ϕ, altitude in meters z, the individual annual values for the 10 years 1903–1912, and the mean for each station increasing from 1.709 at Washington to 1.848 at La Quiaca and 1.861 at Mt. Whitney. These means are plotted on Fig. 53, as $I_{0 \cdot r}$, alongside of μ the vapor pressure of the aqueous vapor in grams per cubic meter μ, and the relative absorption of $\Delta I_{0 \cdot r}$ by $\Delta \mu = 1.00$ gram per 500 meters of vertical path length as will be explained.

In order to determine the maximum value of $I_{0 \cdot r}$ for the pyrheliometer observations, we need to know the bolometer factor

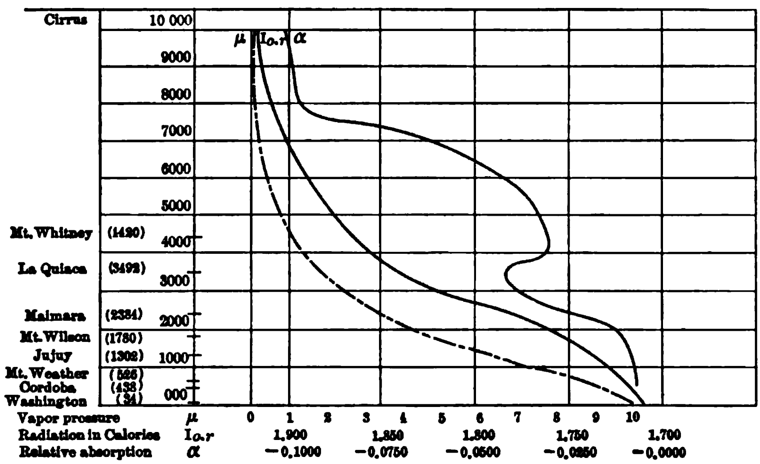


FIG. 53. The relation between the loss of radiation per 1.0 gram per cu. meter for 500 meters, and the ratio between them, a.

F which will supply the amount depleted by line and band absorption. Mr. C. G. Abbot has determined these for Washington, (1.123), and Mt. Wilson (1.094) from a long series of carefully executed bolometer observations. These factors refer the depletion to the adopted curve of a perfect radiator at 6,000° temperature. At present, similar factors are not available for the other stations in this list, so that the data of the table in reduced form are instructive rather than definitive, and it is very important to extend accurate bolometer observations to the other stations employed in making pyrheliometer observations. In section II, the annual values are all reduced to the sea level by the bolometer factors F_1, and in section III to the cirrus level or approximately to the 10,000-meter plane by the factors F_2. The mean value of $I_{0 \cdot r}$ on the sea-level is 1.711 calories, and on the 10,000-meter plane it is 1.918 calories. Section IV contains the mean annual values of p, approximately 0.900 above 1,500 meters, the top of the diurnal convection, and approximately 0.825 below that level. This difference in the transmission factor p is due to the increase of density, ρ, in the lower atmosphere. An example is taken from the data derived from balloon ascensions for summer in the temperate zone, which gives a fair distribution for the year in the low latitude zones.

TABLE 63

RELATIVE EFFICIENCY OF 1 GRAM OF AQUEOUS VAPOR PER CUBIC METER IN ABSORBING THE INCOMING RADIATION PER 500 METERS, MEASURED IN CALORIES

Elevation in Meters	Vapor Pressure		Radiation		Absorption
z	μ	$\Delta\mu$	$I_{0 \cdot r}$	$\Delta I_{0 \cdot r}$	a
10000	0.03	0.02	1.918	−0.002	−0.1000
	0.05	0.02	1.916	− .002	− .1000
9000	0.07	0.02	1.914	− .002	− .1000
	0.09	0.03	1.912	− .003	− .1000
8000	0.12	0.03	1.909	− .003	− .1000
	0.15	0.05	1.906	− .004	− .0800
7000	0.20	0.10	1.902	− .006	− .0600
	0.30	0.13	1.896	− .007	− .0539
6000	0.43	0.17	1.889	− .007	− .0412
	0.60	0.20	1.882	− .007	− .0350
5000	0.80	0.28	1.875	− .009	− .0321
	1.08	0.36	1.866	− .011	− .0310
4000	1.44	0.46	1.855	− .015	− .0326
	1.90	0.60	1.840	− .025	− .0417
3000	2.50	0.79	1.815	− .030	− .0380
	3.29	0.93	1.785	− .025	− .0269
2000	4.22	1.54	1.760	− .016	− .0104
	5.76	1.69	1.744	− .014	− .0083
1000	7.45	1.91	1.730	− .014	− .0073
	9.36	0.87	1.716	− .006	− .0069
000	10.23	....	1.710	....	.....

μ = the vapor pressure in grams per cubic meter.
$\Delta \mu$ = the variation of μ for 500 meters.

$I_{0 \cdot r}$ = the observed radiation in calories per square centimeter per minute, with maximum 1.918 on the 10,000-meter plane.

$\Delta I_{0 \cdot r}$ = the variation in the radiation or absorption per 500 meters.

(687) $\alpha = \frac{\Delta I_{0 \cdot r}}{\Delta \mu}$ the absorption by 1 gram per 500 meters.

The coefficient α multiplied by the observed change in μ for 500 meters gives the amount of radiation which has been absorbed by it. The curve of α shows that there is a gradual diminution of $\Delta I_{0 \cdot r}$ per $\Delta \mu = 1$ gram from 10,000 meters to the sea level, with a maximum of principal absorption, chiefly of short wave lengths, in the stratum 10,000 to 8,000 meters, and a secondary maximum in the stratum 4,000 to 2,000 meters. The laws of selective absorption in the atmosphere for different wave lengths are very complex, and they will require much more research for their complete explanation.

It is quite evident that the curve of $I_{0 \cdot r}$, as derived from the pyrheliometer and bolometer observations at different elevations up to 4,500 meters, at Mt. Whitney, cannot be extended much above 10,000 meters without exaggeration. Furthermore, since the aqueous vapor, upon which the absorption chiefly depends, does not extend above the 10,000-meter, or cirrus levels generally, we are compelled to place the effective source of the radiation observed by the instruments at the cirrus levels 10,000 to 20,000 *m*, rather than on the outermost layer of the atmosphere. Hence, in $\sec z = \frac{m'}{m_0'}$, the unit length m_0' is something like 13,000 meters, rather than the full depth of the atmosphere, and the depletion, or albedo by reflection, is not observed.

3. *The Bolometer and its Energy Spectrum of Radiation*

The bolometer is a complex apparatus, of which the reader can find excellent accounts in the Annals of the Astrophysical Observatory of the Smithsonian Institution. It consists of a siderostat or cœlostat for directing a beam of solar light in a fixed horizontal position, upon a slit which can be adjusted to alter the quantity passing through it; a converging and a collimating mirror to focus the slit upon a prism, which directs the resulting spectrum upon a reflecting flat; a bolometer which consists of a very fine filament of platinum, forming one branch of a delicate Wheatstone balance, including a minute galvanometer in one branch; the lines of the heat spectrum falling upon the bolometer thread modify the current of electricity in the circuits, and deflect the galvanometer which is registered photographically in a manner to act in synchronism with the position of deviation of the spectrum. The movement of the entire spectrum across the bolometer produces a spectrum energy diagram according to the prism, which must be transformed into a normal spectrum of uniform dispersion, and upon this there are very numerous line and band deficiencies of the ordinates due to the selective absorptions that may have occurred. There are numerous coefficients of absorption and reflection in the mirrors and other parts of the apparatus, so that an unusual degree of skill and experience is required for its successful manipulation and the correct interpretation of the resulting ordinates. This more or less defective energy spectrum is to be compared with the full energy spectrum of a perfect radiator, and that of 6,000° C. has been used by the Smithsonian Observatory.

The energy spectrum of a full radiator at a given temperature T may be computed by the Wien-Planck formula, in which J = the energy of radiation at the wave-length λ (λ in microns, $\mu = 0.001$ mm.), $c_1 = 575{,}000$, and $c_2 = 14455$, the adopted constants.

(688) Wien-Planck formula,* $J = c_1 \lambda^{-5} \left(e^{\frac{c_2}{\lambda T}} - 1\right)^{-1}$.

* Transformations of the (*M. K. S.*) and (*C. G. S.*) systems for the mechanical and heat units.

(*C. G. S.*) *System.* 1 *large calorie* = 1 Kilogram degree 1° C. water = 426.8 Kilogram meters = $426.8 \times 1000 \times 100 \times 980.60 = 4.1851 \times 10^{10}$ ergs (Log. 10.62171). 1 *small calorie* = 1 gram degree 1° C. water = 426.8 gram meter = $426.8 \times 100 \times 980.60 = 4.1851 \times 10^{7}$ ergs (Log. 7.62171).

The total energy of radiation is proportional to the fourth power of the temperature and would be expressed by

(689) Stefan formula,

$$J_0 = \sigma T^4 = \frac{c}{w} \cdot 6 \frac{c_1}{c_2^4} T^4 \text{ Watts/cm.}^2 \text{ deg.}^4 \text{ (C. G. S.).}$$

Multiply by 10^7 for ergs/cm.² (C. G. S.) Mech. units, and divide by 4.1851×10^7 for gr. cal./cm.² sec.

c_1 in absolute (C. G. S.) units is about sixty times too large, and depends on the block ($\Delta \lambda$) included in the spectrum measures. The constant is therefore more appropriate to a system with the minute for the unit of time.

The displacement of the maximum ordinate is inversely proportional to the temperature.

(*M. K. S.*) *System.* 1 *large calorie* = 1 Kilogram degree 1° C. = 426.8 kilogram meter = $426.8 \times 1 \times 9.806 = 4.1851 \times 10^3$ (Log. 3.62171). 1 *small calorie* = 1 gram degree 1° C. = 426.8 gram meter = $0.4268 \times 1 \times 9.806 = 4.1851$ (Log. 0.62171).

Transformation Factors. $\frac{\text{Kilogram}}{(\text{meter})^2} = \frac{Gr. \times 10^3}{cm^2 \times 10^4} = \frac{Gr.}{cm^2} \times 10^{-1}$.

$\left(\frac{\text{M. K. S.}}{\text{mech. units}}\right) \times \frac{10 \times 60}{41851000} = \frac{Gr.\ Cal.}{cm.^2 \text{ min.}}$ Factor = 0.000014336 (5.15644–10).

Dimensions and transformation of the gravity equation from (*M. K. S.*) *to* (*C. G. S.*). *Gravity equation.* $g\ (z_1 - z_0) = - \frac{P_1 - P_0}{\rho_{10}} - \frac{1}{2}\ (q_1^2 - q_0^2) - (Q_1 - Q_0)$, (*M. K. S.*). All these terms as computed in the (*M. K. S.*) system are transformed into the (*C. G. S.*) system of mechanical units by the Factor 10^4, as can be tested by substituting the terms dimensionally. Similarly the equations (330) to (337) have the same factor 10000. This equation is transformed from (*M. K. S.*) mechanical units by the factor 1.4336×10^{-5} into $\frac{Gr.\ Cal.}{cm^2.\ min.}$

Dimensions and transformation of the radiation equation.

Radiation equation. $\frac{Q_1 - Q_0}{v_1 - v_0} - P_{10} = \frac{U_1 - U_0}{v_1 - v_0} = K_{10} = c\ T^a$.

All these terms as computed in the (*M. K. S.*) mechanical units, are transformed into (*C. G. S.*) mechanical units by the factor 10, and into $\frac{Gr.\ Cal.}{cm^2.\ min.}$ by the factor $\frac{10 \times 60}{4.1851 \times 10^7} = 0.000014336$.

The transformation of the coefficient in the Stefan Law, $J_0 = \sigma T^4$. The coefficient $\sigma = \frac{3 \times 10^{10}}{4 \times 10^7} \times \frac{6\, c_1}{c_2^4}$, where $\begin{matrix} c_1 = 8\, \pi\, c\, h \\ c_2 = c\, h/k \end{matrix}$ for the Planck constants $h = 6.545 \times 10^{-27}$, $k = 1.3606 \times 10^{-16}$ (*C. G. S.*).

(690) Wien displacement formula, $\lambda_{max} = \frac{2891}{T}$.

The constants used in these formulas have not been found quite the same at all temperatures, especially c_2 seems to have increasing values for higher temperatures, but they serve sufficiently for illustrating the principles now discussed.

If the sun is a body which emits radiation at a given temperature, it will have an efficient energy of radiation per square centimeter per minute at the surface of the emission.

Take R = 694,800 the radius of the sun in kilometers,

D = 149,340,900 the distance to the earth in kilometers

and the effective radiation falling upon the earth's outermost layer of atmosphere, before any reflection or absorption takes place, is found by,

(691) $I_0 = \left(\frac{R}{D}\right)^2 7.90 \times 10^{-11}\, T^4.$

(*C. G. S.*) $c_1 = 8\,\pi\, c\, h = 8 \times 3.14159 \times 3 \times 10^{10} \times 6.545 \times 10^{-27} = 4.93456 \times 10^{-15}$ (−15.69326)

$c_2 = c\,h/k = 3 \times 10^{10} \times 6.545 \times 10^{-27}/1.3606 \times 10^{-16} = 1.4433$ *cm.* (0.15928)

(Mech. units) $\sigma = 3.10^{10} \times 6 \times 4.93456 \times 10^{-15}/4 \times 10^7 \times (1.4433)^4 = 5.1210 \times 10^{-12}$ (−12.70935)

(Heat units) $\sigma = 5.121 \times 10^{-12} \times 10^7/4.1851 \times 10^7 = 1.2236 \times 10^{-12}$ (−12.08764)

(*M. K. S.*) $c_1 = 8\,\pi\, c\, h = 8 \times 3.14159 \times 3.10^8 \times 6.545 \times 10^{-34} = 4.93456 \times 10^{-24}$ (−24.69326)

$c_2 = c\,h/k = 3 \times 10^8 \times 6.545 \times 10^{-34}/1.3606 \times 10^{-23} = 0.014433$ (−2.15928)

(Mech. units) $\sigma = 3 \times 10^8 \times 6 \times 4.93456 \times 10^{-24}/4 \times 10^7 \times (0.014433)^4 = 5.1210 \times 10^{-13}$ (−13.70935)

(Heat units) $\sigma = 5.121 \times 10^{-13} \times 10^7/4.1851 \times 10^8 = 1.2236 \times 10^{-11}$ (−11.08764)

In the Wien-Planck Formula λ in cm. becomes $10^4\,\mu$. Since c_1 is the coefficient of $\lambda^5 = 10^{20}\,\mu$, and c_2 that of $\lambda = 10^4\,\mu$, we shall have,

(*C. G. S.*) c_1 = 493456 (5.69326), c_2 = 14433 (4.15928).

(*M. K. S.*) c_1 = 4934560 (3.69326), c_2 = 14433 (4.15928).

These values are computed for dry air. Kurlbaum's σ for ether, from recent experiments may be taken provisionally:

	C. G. S.	Log.	M. K. S.	Log.
σ in cal./sec.	1.3167×10^{-12}	−12.11948	1.3167×10^{-11}	−11.11948
σ in cal./min.	7.900×10^{-11}	−11.89763	7.900×10^{-10}	−10.89763
σ in ergs/sec.	5.510×10^{-5}	− 5.74119	5.510×10^{-8}	− 8.74119
σ in ergs/min.	3.3063×10^{-3}	− 3.51934	3.3063×10^{-6}	− 6.51934

The following Table 64 gives the solar energy of radiation for certain selected temperatures T. It is computed with the value of $\sigma = 1.316 \times 10^{-12}$ C. G. S., which is not far from the value given by Kurlbaum. In order to make the numbers comparable with the usual convention by which the solar constant is stated in equivalent gram calories for the time-unit of 1 minute, the C. G. S. value of σ is multiplied by 60 in (691).

TABLE 64

EVALUATION OF THE SOLAR RADIATION AT THE MEAN DISTANCE OF THE EARTH FOR SELECTED TEMPERATURES

T	I_0	T	I_0	T	I_0
8000°	7.004	7000°	4.106	6000°	2.216
7900	6.661	6900	3.876	5900	2.072
7800	6.329	6800	3.656	5800	1.935
7700	6.011	6700	3.446	5700	1.805
7600	5.705	6600	3.245	5600	1.682
7500	5.410	6500	3.052	5500	1.565
7400	5.128	6400	2.869	5400	1.454
7300	4.856	6300	2.694	5300	1.349
7200	4.595	6200	2.527	5200	1.250
7100	4.345	6100	2.368	5100	1.157

According to this formula the equivalent "solar-constant" 1.918 would correspond with a temperature 5,787°. At the time the bolometer factors were determined it was supposed that the value $I_{0 \cdot r} = 2.200$ so that 6,000° was assumed to be the proper temperature for the full radiator, with which to compare the actual observed ordinates of the bolometer energy spectrum. Successive improvements in the standardization of the pyrheliometer has, however, reduced that value to 1.918, so that the 5,800° curve would be the proper one to use in determining the bolometer factor at the several stations.

Table 65 contains the evaluation of the Wien-Planck formula of radiation for temperatures from 7,700° to 5,800°, together with the sum of the ordinates from $\lambda = 0.020\ \mu$ to $\lambda = 2.500\ \mu$, the maximum ordinate I_{max}, and the maximum wave-length λ_m, in the displacement formula. Abbot has determined the ordinates in certain arbitrary units for a number of wave-lengths,

TABLE 65

Evaluation of the Wien-Planck Formula of Radiation

$$J = c_1 \lambda^{-5} \left(e^{\frac{c_2}{\lambda T}} - 1\right)^{-1} = \frac{575000}{\lambda^5 \left(10^{\frac{6277 \cdot 4}{\lambda T}} - 1\right)}, \text{ for } c_1 = 575{,}000,\ c_2 = 14{,}455$$

T	7700°	7600°	7500°	7400°	7300°	7200°	7100°	7000°	6900°	6800°	Abbot's Ordinates. 6900°	T_B	I_B
λ = 0.20 μ	3.163	2.796	2.464	2.162	1.892	1.648	1.431	1.241	1.066	0.918	.....	....	.....
.25	6.778	6.139	5.545	4.999	4.489	4.023	3.593	3.199	2.887	2.508	.....	....	....
.30	9.534	8.778	8.066	7.395	6.763	6.170	5.615	5.096	4.612	4.162	0.440	....	....
.35	10.813	10.072	9.367	8.692	8.051	7.439	6.861	6.318	5.795	5.304	2.700	....	....
.40	10.893	10.284	9.600	8.992	8.405	7.844	7.304	6.790	6.298	5.829	4.345	....	....
.45	10.246	9.692	9.153	8.632	8.128	7.639	7.169	6.717	6.280	5.861	6.047	6850	3.765
.50	9.258	8.801	8.356	7.924	7.504	7.094	6.697	6.318	5.941	5.580	6.064	6930	3.944
.55	8.166	7.796	7.434	7.079	6.734	6.398	6.069	5.749	5.438	5.134	5.627	6960	4.018
.60	7.103	6.805	6.511	6.224	5.940	5.663	5.397	5.133	4.875	4.623	5.047	6960	4.018
.70	5.275	5.078	4.886	4.695	4.510	4.325	4.145	3.966	3.791	3.619	3.650	6810	3.678
.80	3.897	3.766	3.638	3.510	3.384	3.260	3.137	3.016	2.896	2.778	2.672	6705	3.456
.90	2.898	2.808	2.721	2.633	2.546	2.460	2.376	2.291	2.208	2.127	2.260	6960	4.018
1.00	2.180	2.117	2.055	1.994	1.933	1.872	1.812	1.753	1.694	1.636	1.664	6850	3.765
1.10	1.662	1.616	1.572	1.527	1.483	1.440	1.397	1.354	1.311	1.269	.....	....	....
1.20	1.288	1.250	1.217	1.185	1.153	1.120	1.088	1.056	1.025	0.994	.....	....	.....
1.30	1.004	0.979	0.955	0.930	0.906	0.886	0.858	0.835	0.811	.787	0.895	7260	4.751
1.40	0.795	.776	.758	.739	.721	.702	.684	.666	.647	.629	.....	....	....
1.50	.637	.623	.608	.593	.579	.565	.551	.537	.522	.509	0.620	7600	5.705
1.60	.516	.504	.493	.481	.470	.459	.447	.436	.426	.414	0.526	7800	6.329
1.70	.421	.412	.403	.394	.385	.377	.368	.359	.350	.341	.....	....	....
1.80	.348	.340	.333	.326	.319	.311	.304	.297	.290	.283	.....	....	....
1.90	.289	.283	.278	.272	.266	.260	.254	.248	.242	.236	.....	....	....
2.00	.242	.237	.232	.228	.223	.219	.214	.209	.204	.199	0.245	7750	6.169
2.10	.205	.201	.197	.193	.189	.185	.181	.177	.173	.169	.....	....	....
2.20	.174	.170	.167	.164	.160	.157	.154	.150	.147	.144	.....	....	....
2.30	.149	.146	.143	.140	.137	.134	.131	.129	.126	.123	.....	....	....
2.40	.128	.125	.123	.120	.118	.116	.113	.111	.109	.106	.....	....	....
2.50	.110	.108	.106	.104	.102	.100	.098	.096	.094	.092	0.043	....	....
Sums	98.167	92.652	87.381	82.327	77.490	72.872	68.448	64.237	60.208	56.369			
Max.	10.974	10.280	9.621	8.997	8.405	7.845	7.315	6.814	6.341	5.895			
λ max.	0.376	0.381	0.386	0.391	0.396	0.401	0.407	0.413	0.419	0.425			

and we have finally decided to reduce them to calories corresponding approximately with 6,900°, and for this purpose have divided them by 1,000. This will be fully explained.

T	6700°	6600°	6500°	6400°	6300°	6200°	6100°	6000°	5900°	5800°
$\lambda = 0.20\ \mu$	0.779	0.662	0.559	0.470	0.393	0.326	0.270	0.222	0.181	0.146
.25	2.209	1.938	1.694	1.474	1.277	1.101	0.945	0.807	0.685	0.579
.30	3.743	3.357	2.999	2.671	2.370	2.095	1.844	1.616	1.411	1.226
.35	4.844	4.411	4.006	3.626	3.273	2.944	2.639	2.358	2.098	1.859
.40	5.382	4.958	4.555	4.177	3.817	3.478	3.160	2.862	2.584	2.322
.45	5.458	5.073	4.705	4.353	4.017	3.698	3.395	3.108	2.838	2.588
.50	5.232	4.896	4.574	4.264	3.965	3.681	3.408	3.147	2.898	2.661
.55	4.841	4.557	4.280	4.014	3.757	3.509	3.270	3.041	2.821	2.610
.60	4.378	4.140	3.909	3.683	3.465	3.253	3.049	2.851	2.660	2.477
.70	3.451	3.287	3.125	2.968	2.815	2.664	2.517	2.375	2.236	2.101
.80	2.662	2.548	2.436	2.326	2.218	2.112	2.009	1.907	1.807	1.710
.90	2.046	1.965	1.886	1.808	1.732	1.657	1.583	1.509	1.438	1.367
1.00	1.577	1.521	1.464	1.409	1.353	1.298	1.245	1.191	1.139	1.089
1.10	1.226	1.185	1.144	1.103	1.063	1.028	0.983	0.944	0.906	0.869
1.20	0.963	0.932	0.901	0.871	0.840	0.811	.782	.752	.724	.695
1.30	.763	.740	.717	.694	.672	.649	.626	.604	.582	.560
1.40	.611	.593	.576	.558	.541	.523	.506	.489	.472	.455
1.50	.495	.481	.467	.453	.439	.426	.413	.399	.386	.372
1.60	.408	.392	.381	.371	.360	.349	.338	.328	.318	.307
1.70	.332	.323	.315	.306	.297	.289	.280	.272	.263	.255
1.80	.276	.268	.262	.255	.248	.241	.234	.227	.220	.214
1.90	.231	.225	.219	.214	.207	.202	.197	.191	.185	.180
2.00	.196	.189	.185	.180	.176	.171	.166	.162	.157	.152
2.10	.165	.161	.157	.153	.149	.145	.141	.138	.134	.130
2.20	.140	.137	.134	.131	.127	.124	.121	.118	.114	.111
2.30	.121	.118	.115	.113	.109	.107	.104	.101	.099	.096
2.40	.104	.102	.099	.097	.095	.092	.090	.088	.085	.083
2.50	.090	.088	.086	.084	.082	.080	.078	.076	.074	.072
Sums.	52.718	49.247	45.950	42.826	39.857	37.038	34.392	31.873	29.515	27.281
Max.	5.474	5.077	4.704	4.354	4.024	3.714	3.424	3.153	2.899	2.661
λ max.	0.432	0.438	0.445	0.452	0.459	0.466	0.474	0.482	0.490	0.498

Fig. 54 contains the diagrams of the energy curves for 7,700°, 6,900°, 5,800°, and Abbot's observed ordinates, which closely agree with the 6,900° curve from wave-length $\lambda = 0.50\ \mu$ to $\lambda = 1.00\ \mu$. It is seen that there is a heavy depletion from $\lambda = 0.00$ to $0.30\ \mu$ or $0.35\ \mu$, a slight excess from $\lambda = 0.50\ \mu$ to $0.70\ \mu$, a slight deficiency from $\lambda = 0.70\ \mu$ to $1.00\ \mu$, and a rapid increase in the ordinates to agree with those of the 7,700° curve for $\lambda = 1.50\ \mu$ to $2.50\ \mu$. Whether the matching of the observed with the computed ordinates is made for the curves 6,500°, 6,600°, 6,700°, 6,800°, 6,900°, 7,000°, the preceding description of results holds true, but the best coincidence seems to be for 6,900°.

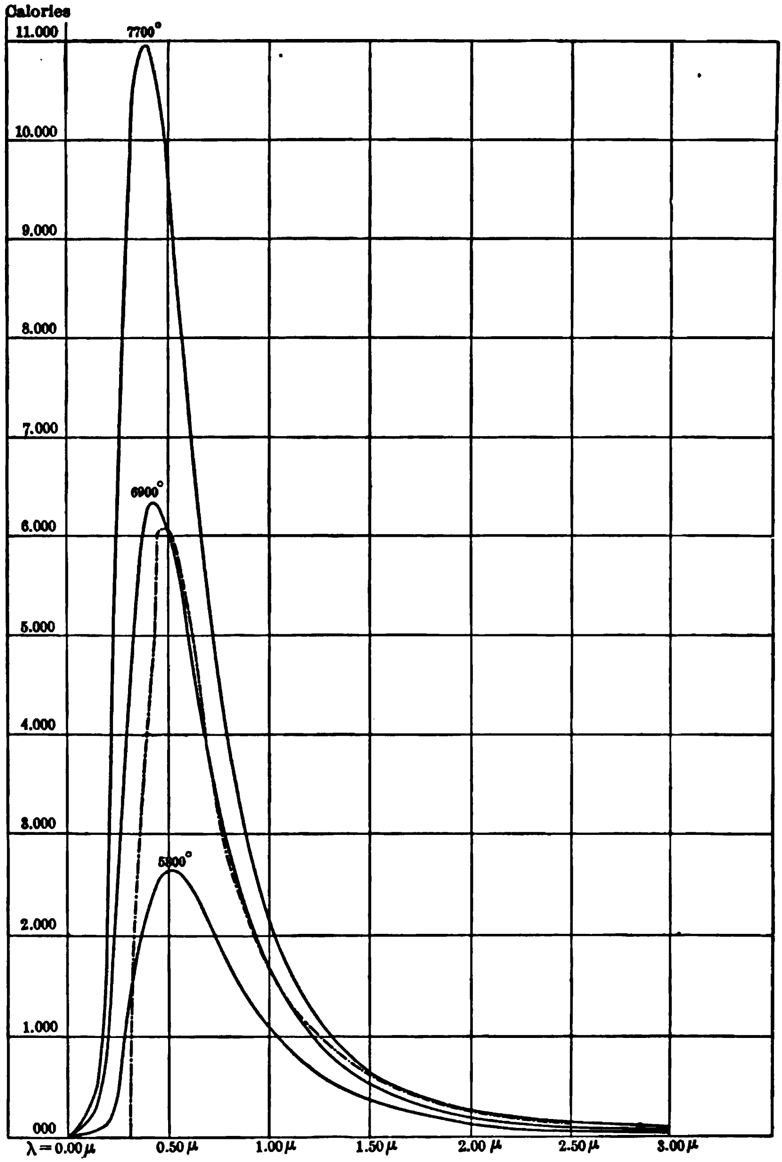


FIG. 54. Observed bolometer ordinates in their relation to the perfect radiation spectrum at different temperatures.

Now this corresponds with 3.876 for the solar constant at the earth, instead of 1.918 as given for the maximum from pyrheliometer observations, and this requires $\frac{3.876}{1.918} = 2.02$ times as many calories to fall upon the surface of the outer layer of the atmosphere as upon the pyrheliometers, so that at least 1.958 calories must have been reflected back into space as albedo, which would therefore be about 51 per cent of the radiation of the sun received at the earth. If the bolometer ordinates can be more accurately determined this ratio may be definitely found; if the ratio varies from year to year at the same, or at several stations, the proper distribution between the variability of the solar radiation and the variability of the terrestrial reflection can be further discussed. At present, one must be very conservative in attributing to solar variation the entire apparent variations of the radiation measured by the pyrheliometer and the bolometer.

The remarkable fact appears to be established by Abbot's bolometer ordinates, Table 65, that these ordinates do not correspond with any single temperature of emission. By comparing the Abbot ordinate at any given wave-length with the ordinates computed at different temperatures for the same wave-length, it is not difficult to interpolate for a temperature of emission that would produce the ordinate. Thus, they are 6,850° at $\lambda = 0.40\ \mu$, 6,960° at $\lambda = 0.55\ \mu$, 7,260° at $\lambda = 1.30\ \mu$, 7,800° at $\lambda = 1.60\ \mu$ to 2.00 μ. On the face of it, the Abbot ordinates range through 1,000° temperature, 6,700° to 7,700°, and this may have several interpretations. (1) The solar envelope may consist of layers of different temperatures, 7,700° at the photosphere, which would be the general source of emission for all wave-lengths, gradually diminishing to 6,700° at the top of the chromosphere, or possibly the inner corona, in which envelope there is gradual selective absorption of certain wave-lengths, so that the effective emission of the sun to space is very complex, and corresponds with the observed bolometer spectrum; (2) a similar selective depletion of a uniform spectrum of 7,700°-energy may occur in the middle and lower region of the earth's atmosphere, so that

TABLE 66

VALUES OF $\frac{d^2 Q}{d z^2}$ IN THE RADIATION FORMULA $\frac{d Q}{d t} = k^2 \frac{d^2 Q}{d z^2}$ MECHANICAL UNITS (M. K. S.) PER 1,000 METERS

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°	
					CASE I. Low temperatures in the isothermal region						
19000	−282.6	−298.6	−306.5	−307.4	−306.3	−324.3	−341.4	−329.8	−331.1	−319.7	Isothermal region I_1
18000	−296.2	−297.7	−315.2	−320.5	−324.2	−348.7	−329.8	−349.4	−345.3	−331.6	
17000	−309.5	−316.2	−319.9	−339.5	−327.3	−317.6	−362.8	−337.2	−341.7	−343.6	
16000	−322.2	−325.2	−329.4	−336.5	−337.4	−356.2	−336.9	−285.0	−306.8	−312.2	
15000	−337.5	−344.8	−344.4	−336.6	−357.5	−344.2	−265.5	−202.6	−231.7	−235.8	
14000	−345.8	−351.6	−358.7	−361.2	−326.1	−315.0	−151.5	−162.4	−167.9	−169.8	Absorption and reflecting region R_1
13000	−269.6	−275.2	−253.0	−273.6	−301.6	−200.5	− 92.9	−142.3	−153.3	−148.7	
12000	−196.3	−194.3	−179.1	−175.6	−176.4	−109.9	− 49.0	− 98.6	−101.8	− 99.6	
11000	−189.6	−191.1	−144.5	−103.3	− 70.8	+ 25.9	− 23.9	− 65.7	− 71.1	− 46.4	
10000	−173.3	−181.9	−123.8	− 83.9	− 57.9	− 25.1	− 23.0	− 71.5	− 47.9	− 49.1	

CASE II. High temperatures in the isothermal region

19000	−244.9	−250.1	−255.7	−271.1	−269.9	−285.9	−360.5	−325.2	−288.2	−248.1	
18000	−244.9	−266.1	−275.3	−274.9	−282.3	−303.9	−354.0	−367.4	−310.4	−241.2	
17000	−276.6	−271.7	−276.4	−283.5	−295.6	−301.9	−344.5	−385.9	−277.7	−259.0	Isothermal region I_2
16000	−281.9	−288.8	−293.6	−284.2	−309.2	−321.0	−341.4	−356.8	−303.2	−243.8	
15000	−293.7	−273.4	−308.8	−336.5	−335.5	−313.9	−270.5	−238.0	−253.6	−207.3	
14000	−306.4	−320.8	−321.6	−332.9	−327.1	−311.2	−284.1	−197.0	−176.0	−169.9	
13000	−321.7	−342.7	−347.6	−361.1	−378.5	−353.3	−232.5	−144.5	−148.0	−142.3	
12000	−335.3	−351.0	−383.8	−395.6	−426.5	−360.7	−112.7	− 86.7	− 99.9	−103.4	Absorption and reflecting region R_1
11000	−351.9	−328.6	−228.7	−620.2	−646.9	−504.3	−129.8	− 30.0	− 36.7	− 52.2	
10000	−262.2	−277.8	−235.7	+237.6	+358.6	+474.3	+187.2	− 79.4	− 59.1	− 52.3	
9000	−167.2	−162.9	−144.5	− 96.0	− 57.7	− 51.4	− 92.0	− 85.1	− 77.8	− 76.0	
8000	−168.4	−164.9	−136.2	− 78.2	− 85.5	−151.2	−144.6	−114.1	−113.0	−109.6	
7000	−171.9	−152.1	−148.7	−143.3	−120.8	−183.9	−160.2	−149.1	−144.8	−147.6	
6000	−192.0	−163.1	−166.3	−146.3	−151.1	−152.4	−155.7	−152.4	−162.3	−177.5	
5000	−215.0	−180.4	−188.3	−163.8	−162.8	−145.6	−129.5	−156.9	−167.8	−173.2	Convectional region C
4000	−226.0	−202.1	−196.3	−134.4	−159.2	−161.9	−130.5	−158.6	−168.7	−163.8	
3000	−233.9	−211.4	−216.4	−139.7	−153.7	−169.9	−173.9	−182.4	−170.7	−163.8	
2000	−259.9	−239.0	−231.5	−210.0	−168.6	−140.3	−146.9	−125.5	−103.2	− 77.7	
1000	−140.4	−125.2	−125.8	−109.7	− 91.9	− 62.1	− 45.5	− 34.6	− 18.5	− 24.4	Absorption region R_2
000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	
Mean I_1	−309.1	−315.6	−318.2	−325.0	−325.8	−343.3	−327.3	−325.4	−331.2	−326.8	
" I_2	−292.0	−297.1	−292.7	−318.7	−328.1	−318.9	−312.5	−334.7	−286.6	−248.0	
" C	−204.3	−184.4	−178.5	−156.3	−152.7	−157.9	−148.8	−148.5	−147.2	−144.8	
Ratio I_1/C	.....	.....	.....	2.08	2.13	2.17	2.19	2.18	2.25	2.26	
" I_2/C	.....	.....	.....	2.04	2.15	2.02	2.09	2.24	1.95	1.71	
Mean	.....	.....	.....	2.06	2.14	2.10	2.14	2.21	2.10	1.99	2.11

the observed spectrum is only the remainder that has escaped from the terrestrial depletion in the lower levels; (3) a mixture of these two cases may be more closely related to the facts, but if so, there will be great difficulty in disentangling the elements. Corresponding with these temperatures we may compute the value of the radiation at the earth, on the first supposition, that a depleted radiation escapes from the solar envelope, and we find 3.765 calories for $\lambda = 0.45\ \mu$, about 4.000 for $\lambda = 0.50\ \mu$ to $0.60\ \mu$, a lower variable amount from $\lambda = 0.70\ \mu$ to $1.00\ \mu$, 4.751 at $\lambda = 1.30\ \mu$, 5.705 at $\lambda = 1.50\ \mu$, 6.329 at $\lambda = 1.60\ \mu$, 6.169 at $\lambda = 2.00\ \mu$. The evidences indicate that the solar intensity of radiation at the earth is from 4.00 to 6.00 calories per square centimeter per minute.

The isothermal region radiates 2.11 times as much heat as does the convectional region. The isothermal region is separated from the convectional region by a wedge-shaped layer, which

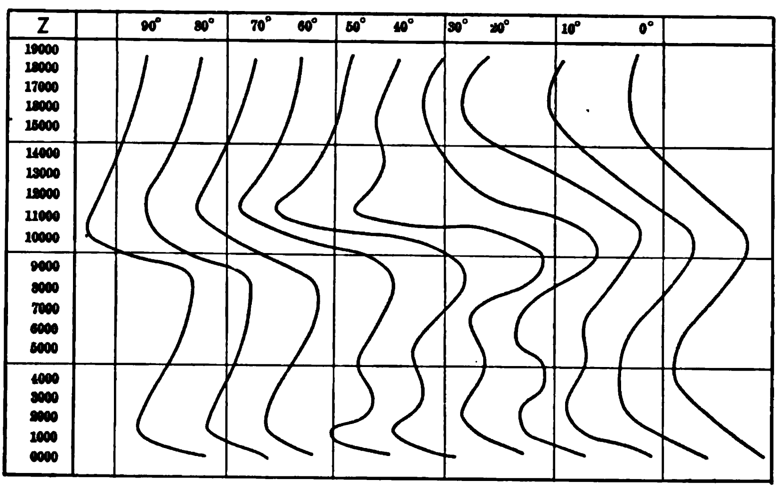


FIG. 55. The courses of the relative radiation.

partly reflects the incoming ray and partly absorbs it. The remainder proceeds through the convectional region, suffering depletion by minor absorptions to the cumulus layer, where it is more rapidly absorbed, and the balance reaches the sea level or land areas. The outgoing ray experiences a similar complex series of absorptions and reflections. The absorption is accompanied by transformation of energy into ionization currents in the cirrus layer and in the cumulus layer, the electric streams of the former flowing to the poles cause the auroras, and by induction the aperiodic variations in the magnetic field; the ionization currents of the cumulus region are controlled by the diurnal convection circulation, and induce the diurnal variations of the magnetic field.

TABLE 67

APPROXIMATE MEAN VALUES OF $\frac{d^2 Q}{d z^2}$ IN THE RADIATION EQUATION $\frac{d Q}{d t} = k^2 \frac{d^2 Q}{d z^2}$ PER 1000 METERS

z	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
19000	−245	−250	−256	−271	−270	−286	−310	−325	−288	−248
18	−245	−266	−275	−275	−282	−304	−354	−367	−310	−241
17	−277	−272	−286	−283	−296	−302	−345	−386	−308	−259
16	−282	−289	−294	−284	−309	−321	−341	−357	−303	−244
15000	−294	−303	−309	−317	−336	−314	−320	−288	−254	−207
14	−306	−321	−322	−333	−327	−311	−284	−196	−176	−170
13	−322	−343	−348	−361	−379	−353	−233	−145	−148	−142
12	−335	−352	−384	−396	−427	−361	−113	− 87	−100	−103
11	−352	−329	−309	−200	−180	−110	− 49	− 40	− 37	− 52
10000	−262	−278	−236	−182	−108	− 25	− 23	− 79	− 59	− 52
9	−167	−163	−144	− 96	− 58	− 51	− 92	− 85	− 78	− 76
8	−168	−165	−136	− 78	− 86	−151	−145	−114	−113	−110
7	−172	−152	−149	−143	−121	−154	−160	−149	−145	−148
6	−192	−163	−166	−146	−151	−152	−156	−152	−162	−178
5000	−215	−180	−188	−164	−163	−146	−130	−157	−168	−173
4	−226	−202	−196	−134	−159	−162	−131	−159	−169	−164
3	−234	−211	−216	−140	−154	−170	−174	−182	−171	−164
2	−260	−239	−232	−210	−189	−140	−147	−126	−103	− 78
1	−140	−125	−126	−110	− 92	− 62	− 46	− 35	− 19	− 24
0000	...	...	...	...	...	...	...	...	...	...

The mean values of $\frac{d^2 Q}{d z}$ are collected in Table 67 and plotted in Fig. 55. The radiation increases downwards to a maximum at the 12,000-meter level, then there is a large increase in the heat supply by absorption in the cirrus region, then an increase in radiation to the cumulus level, and a second supply of heat is at hand extending to the surface.

The Values of $\frac{d^2 Q}{d z^2}$ *in the Radiation Equation*

4. There is another important argument leading to the same conclusion, that the so-called "solar-constant" exceeds 4.00 calories, derived from the thermodynamic computations described in Chapter III, the results of the same being more fully published in Bulletin No. 3, Oficina Meteorologica Argentina, 1912. The value of $(Q_1 - Q_0)$, the loss of heat by radiation for every 1,000 meters difference of level, and for every 10 degrees of latitude from the equator to the pole, was derived from the temperature data of the balloon ascensions. Table 66 contains the second differences, $\int \frac{d^2 Q}{d z^2}$, which are found in the radiation formula,

$$(692) \quad \frac{d\,Q}{d\,t} = k^2 \frac{d^2 Q}{d\,z^2},$$

and from which, having once obtained the coefficient k^2, the true radiation $\frac{d\,Q}{d\,t}$, loss of heat in the unit time, may be computed. For our present purpose we need only admit that the rate of radiation is proportional to the second differences $\Delta^2 (Q_1 - Q_0)$ per thousand meters, to be able to make important inferences.

The two cases of low I_1 and high I_2 temperatures in the isothermal region are considered in connection with average conditions in the convectional region C; between the isothermal and the convectional region there is a region of absorption and reflection of the radiation R, wedge-shaped, about 6,000 meters deep over the equator, and thin or vanishing in the arctic zone; below the convectional region there is a second stratum of absorption and reflection, less than 2,000 meters deep, and occupying the region of the specific diurnal convection R_2. We are not now concerned with the absorption regions R_1, R_2, but only with the radiation regions I_1, I_2, C. Taking the mean values of $\Delta^2 (Q_1 - Q_2)$ in these regions, respectively, and the ratios I_1/C and I_2/C, together with the means, it results that about 2.11 times as much radiation is passing through the isothermal region as passes through the convectional region. This can only signify that of the incoming radiation 52 per cent is reflected before reaching the surface of the earth, and that a nearly equal amount returns to space as albedo. If we again admit that the 1.918 calories determined as the maximum passes below the cirrus levels to the lower stations, it follows that,

$$2.11 \times 1.918 = 4.047 \text{ calories,}$$

is the approximate average value of the "solar-constant." This confirmation of the results from the bolometer ordinates also strengthens the conclusion that the solar temperature of emission is about 6,900°. Since $I_{0 \cdot r} = 1.918$ calories, and this corresponds nearly with the temperature 5,800° in Table 65, the sum of the ordinates for a full radiator at 5,800°, summing them from $\lambda = 0.20\,\mu$ to $\lambda = 2.50\,\mu$, is about 27.28. Now, this sum is proportional to the area of the energy spectrum between these wave lengths, beyond which limits the energy not included is very small in amount, and since we require to employ 2.11 times this energy,

$$2.11 \times 27.28 = 57.56,$$

it is seen that the sum of the ordinates equal to this amount lies between 6,800° and 6,900° solar temperature. If an allowance be made for selective absorptions in the several zones of the earth's atmosphere, as in the cirrus region, it is evident that these general conclusions may be extended to many important special researches.

We have, therefore, besides the negative argument that the Bouguer formula is indeterminate, so far as fixing the level from which the observed radiation proceeds to the surface, the three positive arguments that the solar constant is 4.00 calories upward, and the temperature of the solar photosphere, 6,900°

upward, namely, (1) The maximum 1.918 calories on the 10,000-meter plane, from stations at different elevations above the sea level. (2) The form of the bolometer ordinate curves of the energy spectrum for 6,900° solar temperature. (3) The thermodynamic ratio 2.11 of the radiation in the isothermal and convectional regions. Compare in addition Chapter VII.

The Measures of the Ionization of the Atmosphere

There are two types of instruments for measuring the ionization of the atmosphere, (1) the Elster and Geitel apparatus for the coefficient of dissipation of electric charges from a body connected with an electroscope, (2) the Ebert ion-counter with fixed capacity and variable charge, and the Gerdien ion-counter with fixed charge and variable capacity. The former measures the coefficient of dissipation λ, and the two latter measure the component parts of λ, namely, the number of ions per cubic centimeter of air n, and the velocity of motion of the ions which dissipate the charge on the body attached to the electroscope. For each kind of electricity, we have,

$$\lambda = (e\, n_+ u_+ + e\, n_- u_-)$$

where $e = 3.4 \times 10^{-10}$, the constant charge on the single atom. These subjects have been discussed in many papers, and the results are collected in: *Die Atmosphärische Electrizität, H. Mache* und *E. V. Schweidler*, Braunschweig, 1909; *Die Luftelektrizität, Albert Gockel*, Leipzig, 1908; *Electricité Atmosphérique*, Observatoire de l'Ebre, 1910, par le *P. J. Garcia Mollá, S.J.*

There is a fundamental discrepancy, amounting to about 300 per cent, between the resulting values of λ observed and computed by the two methods indicated, and as it is our purpose to come to this problem, we shall as briefly as possible summarize the formulas and the conditions of the instruments leading to these conclusions. The elementary electrostatic formulas can be verified by reference to numerous treatises, of which one of the most convenient is, *Elements of the Mathematical Theory of Electricity and Magnetism*, by *J. J. Thomson*, Cambridge, 1895.

It is not necessary further to explain the individual formulas which follow by simple transformations from the elementary conditions that have been stated.

Notation and Elementary Relations

The sign + signifies repulsion along the line joining two charges of the same kind $Q\,Q'$.

Q = charge on a point, Q_1 = charge on unit line, Q_2 = charge on unit area.

(693) Force $= F = \frac{Q\,Q'}{r}$. F_n = the normal, and F_t = the tangential components.

(694) Total intensity $= I = \Sigma F_n S = \int F_n\, d\, S = 4\,\pi\, Q$ on the closed surface S.

The induction is zero when the charge is outside S.

Special applications to given surfaces for unit lengths.

	Sphere	*Cylinder*
	Total force $F = F_s \,.\, 4\,\pi\, r^2 = 4\,\pi\, Q$	$F = F_c\, 2\,\pi\, r = 4\,\pi\, Q_1$
(695)	$F_s = \frac{Q}{r^2}$	$F_c = \frac{2\,Q_1}{r}$

Infinite Plane

$$F = F_p\, S = 4\,\pi\, Q_2 \,.\, S$$

$$F_p = 4\,\pi\, Q_2$$

The difference of potential $\Delta\, V$ = the work done in moving the unit charge against the electric field of intensity F_n through the distant $\Delta\, r$.

(696) Work. $$W = \int_1^{r_2} F_n\, d\, r = F_n\,(r_2 - r_1) = (V_1 - V_2)$$

for the unit charge.

$$W = \frac{Q}{r_2} - \frac{Q}{r_1} = \frac{r_1 - r_2}{r_1\, r_2} \,.\, Q = (V_1 - V_2)\, Q$$

for the charge Q.

(697) Force. $F_n = -\frac{(V_2 - V_1)}{r_2 - r_1} = -\frac{dV}{dr}$. Mechanical force $F = \frac{1}{2} F_n Q$.

(698) Work. $W = \int F_n \, dr = -\int \frac{Q}{r^2} \, dr = \frac{Q}{r} = (V_1 - V_2)$ for the sphere.

(699) Inner energy. $U = \frac{1}{2} W = \frac{1}{2} Q (V_1 - V_2) = \frac{1}{2} C (V_1 - V_2)^2 = \frac{1}{2} p Q^2$, for $p = \frac{1}{C}$.

The following quantities, Q, D, Π, σ, n, are synonyms for the same physical process in a dielectric, and differ only by coefficients.

E = the electric force, c = the specific conductive capacity,
H = the magnetic force, μ = the magnetic permeability,
$D = cE$ = the electric displacement; $B = \mu H$ = the magnetic induction. $\Pi = \frac{1}{4\pi} F_n$ = the polarization = σ the surface density = Q the charge.

$$(670)\quad U = \tfrac{1}{2} W = \tfrac{1}{2} D\,E = \tfrac{1}{2} c E^2 = \tfrac{1}{2}\frac{D^2}{c} = \tfrac{1}{2} E (V_1 - V_2).$$

In a medium other than air the specific inductive capacity is K. The number of Faraday lines = n.

$$(671)\quad F_n = E = \frac{4\pi}{K} D = \frac{4\pi}{K}\sigma = \frac{4\pi}{K}\Pi = \frac{4\pi}{K} n.$$

$$(672)\quad U_l = \tfrac{1}{2} \cdot \frac{K}{4\pi} F_n^2 = \frac{K}{2} \cdot n F_n = \text{the longitudinal tension.}$$

$$(673)\quad U_t = \tfrac{1}{2} \cdot \frac{K}{4\pi} \cdot F_t^2 = \frac{K}{2} \cdot n F_t = \text{the transversal tension.}$$

A very complete collection of electrical and magnetic formulas may be found in Bulletin I, U. S. W. B., 1902, "Eclipse Meteorology and Allied Problems."

Electrostatic Relations per Unit Length

$$Q = C\,V = \sigma\,S$$

Formula	Sphere	Cylinder	Plane
(674) Charge $Q = C\,V$	$Q = r\,V$	$Q_1 = \dfrac{V}{2 \log_e r}$	$Q_2 = \dfrac{V}{4\pi r}$
(675) Potential $V = Q/C$	$V = \dfrac{Q}{r}$	$V = 2\,Q_1 \log_e r$	$V = 4\pi\,r.\,Q_2$
(676) Capacity $C = Q/V$	$C = r$	$C = \dfrac{1}{2 \log_e r}$	$C = \dfrac{1}{4\pi r}$
(677) Potential coefficient $p = V/Q$	$p = \dfrac{1}{r}$	$p = 2 \log_e r$	$p = 4\pi r$
(678) Force $F = -\dfrac{dV}{dr}$	$F_s = +\dfrac{Q}{r^2}$	$F_c = +\dfrac{2\,Q_1}{r}$	$F_p = 4\pi\,Q_2$
(679) Work $U = \frac{1}{2} Q\,V$ $U = \frac{1}{2} p\,Q^2 = \frac{1}{2} C\,V^2$	$U = \frac{1}{2}\dfrac{Q^2}{r}$	$U = Q_1^2 \log_e r$	$U = 2\pi\,r\,Q_2^2$

Formula	Two Concentric Spheres	Two Coaxial Cylinders	Two Parallel Planes
(680) Potential $V_1 - V_2$	$V_1 - V_2 = Q\,\dfrac{r_2 - r_1}{r_1 r_2}$	$V_1 - V_2 = 2\,Q_1 \log_e \dfrac{r_1}{r_2}$	$V_1 - V_2 = 4\pi\,Q_2\,(r_1 - r_2)$
(681) Capacity $C = \dfrac{Q}{V_1 - V_2}$	$C = \dfrac{r_1 r_2}{r_2 - r_1}$	$C = \dfrac{1}{2 \log_e \dfrac{r_1}{r_2}}$	$C = \dfrac{1}{4\pi\,(r_1 - r_2)}$
(682) Charge $Q = C\,(V_1 - V_2)$	$Q = \dfrac{r_1 r_2}{r_2 - r_1}(V_1 - V_2)$	$Q_1 = \dfrac{1}{2 \log_e \dfrac{r_1}{r_2}}(V_1 - V_2)$	$Q_2 = \dfrac{1}{4\pi\,(r_1 - r_2)}(V_1 - V_2)$
(683) Potential coefficient $p = \dfrac{V_1 - V_2}{Q}$	$p = \dfrac{r_2 - r_1}{r_1 r_2}$	$p = 2 \log_e \dfrac{r_1}{r_2}$	$p = 4\pi\,(r_1 - r_2)$
(684) Work $U = \frac{1}{2} Q\,(V_1 - V_2)$ $U = \frac{1}{2} P\,Q^2 = \frac{1}{2} C\,(V_1 - V_2)^2$	$U = \frac{1}{2}\dfrac{r_2 - r_1}{r_1 r_2} Q^2$	$U = \log_e \dfrac{r_1}{r_2} Q_1^2$	$U = 2\pi\,(r_1 - r_2)\,Q_2^2$
(685) Surface density $\sigma = \dfrac{Q}{S}$	$\sigma = \dfrac{Q}{4\pi\,(r_1^2 - r_2^2)}$	$\sigma = \dfrac{Q_1}{2\pi\,(r_1 - r_2)\,l}$	$\sigma = \dfrac{Q_2}{S}$
(686) Force $F = 4\pi\sigma$	$F = \dfrac{Q}{(r_1^2 - r_2^2)}$	$F = \dfrac{2\,Q_1}{(r_1 - r_2)\,l}$	$F = -\dfrac{4\pi\,Q_2}{S}$
Quantity neutralized in the unit time (687) $\int i\,dS = 4\pi\lambda\,Q = a\,Q$ $= 4\pi\,[(\epsilon n u)_+ + (\epsilon n u)_-]\,Q$ $= 4\pi\lambda\,.\,C\,V = 4\pi\lambda\,.\,\sigma\,S$	$\int i\,dS = 4\pi(\epsilon n u)\,Q$	$\int i\,dS = 4\pi(\epsilon n u)\,Q_1$	$\int i\,dS = 4\pi\,(\epsilon n u)\,Q_2$

Derivation of Coulomb's Law of Dissipation

(688) $$\int i\,dS = 4\,\pi\,\lambda\,.\,Q = 4\,\pi\,\lambda\,.\,\sigma\,S = \lambda.\,F.\,S = a\,Q.$$

(689) $\int i\,dS = -\frac{dQ}{dt}$. The loss of charge with the time dt.

(690) $4\pi\lambda . Q = -\frac{dQ}{dt}$. Combining (689) and (690) in Coulomb's Law,

(691) $4\pi\lambda . dt = -\frac{dQ}{Q}$; $4\pi\lambda(t_1 - t_0) = -\log_e \frac{Q_1}{Q_0}$;
$Q_1 = Q_0 a^{-4\pi\lambda(t_1 - t_0)}$.

(692) $V_1 = V_0 e^{-a(t_1 - t_0)}$.

(693) $a(t_1 - t_0) = C.\ \text{log. nat}\ \frac{V_0}{V_1} = \frac{1}{M} . C \log \frac{V_0}{V_1}$

Conduction Ionization Currents

Number of ions per cu. cm. n_+ n_-

* Charge in E. S. U. e_+ e_- $e = 3.4 \times 10^{-10}$.

Velocity of ions in cm/sec at $\frac{1\ \text{E.S.U.}}{\text{cm.}}$, u_+ u_-

(694) Conductivity = the positive current through 1 cm² in unit time, $\lambda = (e_+ n_+ u_+) + (e_- n_- u_-)$.

Example of an Insulated Charged Sphere of Radius r

(695) Charge. $Q = CV = rV. = \sigma S$.

(696) Surface density. $\sigma = \frac{Q}{4\pi r^2} = \frac{Q}{S}$. S = the surface of the sphere.

(697) Potential. $V = \frac{Q}{r} = 4\pi r . \sigma$.

(698) Capacity. $C = r = \frac{Q}{V}$.

(699) Force. $F_n = \frac{Q}{r^2} = 4\pi\sigma$, acting on the unit quantity at distance r.

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* Result of many experiments. Compare Table 95.

(700) Ion current $i = \lambda F = [(e\,n\,u)_+ + (e\,n\,u)_-]\,\frac{Q}{r^2}$

from the unit area.

(701) On whole sphere $= 4\,\pi\,r^2.\,\lambda\,F_n = 4\,\pi\,[(e\,n\,u)_+ + (e\,n\,u)_-]\,Q. = \int i\,d\,S.$

This ionization current neutralizes the charge of the body at a certain rate by $4\,\pi\,(e\,n\,u)_+\,Q$ acting outward, and $4\,\pi\,(e\,n\,u)_-\,Q$ acting inward, as to the surface.

The dissipation coefficient λ is the measure of the rate of the neutralizing the charge.

(702) $\lambda_+ = 4\,\pi\,(n\,e\,u)_-$ neutralized negative charge.

$\lambda_- = 4\,\pi\,(n\,e\,u)_+$ neutralized positive charge.

(703) Quantity neutralized $= \lambda\,Q = \lambda\,r\,V = 4\,\pi\,\lambda\,r^2\,\sigma.$

Large bodies neutralize faster than small bodies.

The neutralization amount is proportional to σ.

The Coefficient of Electrical Dissipation of the Atmosphere λ

The Elster and Geitel apparatus consists of an electroscope, whose capacity by itself is C, and a small dissipating body, generally a copper cylinder 10 cms. long and 5 cms. in diameter, whose capacity by itself is K. The dissipating body charged with the quantity of electricity Q, in the time $d\,t$, loses the quantity $-\,d\,Q$, by the formula for voltage,

(704) $$-\frac{d\,Q}{d\,t} = -\,(C + K)\,\frac{d\,V}{d\,t}.$$

Since it is required to know the rate of dissipation from the cylinder alone, where,

(705) $Q = K\,V,$

we construct the Coulomb law, by division,

(706) $$4\,\pi\,\lambda = a = -\frac{1}{Q}\,\frac{d\,Q}{d\,t} = -\,\frac{C + K}{K}\,\frac{d\,V}{d\,t}.$$

Multiply by $d\,t$ and integrate for common logarithms,

$$(707)\quad a\,(t_1 - t_0) = \frac{C+K}{K}\cdot\frac{1}{M}\cdot\log\frac{V_0}{V_1}.$$

This is the formula for dissipation from the body to the free air, provided there is no leakage into the interior of the electroscope. This is expressed, from parallel reasoning, for C and K separately,

$$(708)\quad a\,(t_1 - t_0) = \frac{C}{K}\,\frac{1}{M}\log\frac{V_0'}{V_1'}.$$

Combining these two parts into one formula,

$$(709)\quad 4\,\pi\,\lambda = a = \frac{1}{t_1 - t_0}\cdot\frac{C+K}{K}\cdot\frac{1}{M}\cdot\left(\log\frac{V_0}{V_1} - \frac{C}{C+K}\log\frac{V_0'}{V_1'}\right).$$

If the instrument is charged with the electroscope alone for V_0' at the time t_0, then the V_1' is to be read at the time t_1, generally an interval of 15 minutes for $(t_1 - t_0)$. When the insulation is good this correction is small, and the compensation can be made by $(V_1) = V_1 + \Delta\,V_1$. It is customary to express the constant a in terms of percentage of loss of charge per minute, so that,

$$(710)\quad a' = \frac{100}{15}\cdot\frac{K+C}{K}\cdot\frac{1}{M}\cdot\log\frac{V_0}{(V_1)}.$$

Finally, the general formula for λ becomes in E. S. U.,

$$(711)\quad \lambda = \frac{a}{4\,\pi} = \frac{a'}{4\,\pi\times 60\times 100} = \frac{100}{4\,\pi\times 60\times 100}\cdot\frac{C+K}{K}\cdot\frac{1}{M}\cdot\log\frac{V_0}{(V_1)} = (e\,n_+\,u_+ + e\,n_-\,u_-)\,300.$$

The Ebert ion counter consists of an electroscope to which a condenser is attached as an integral part, and through this a known volume of air A in cubic centimeters per second is drawn by a revolving turbine. The capacity C of this combination is known. In the axis of the long cylinder of radius r, a small cylinder of radius r_2 and length l is supported by an insulating rod connecting with the electroscope. If the apparatus runs for a given time $(t_1 - t_0)$ while the small cylinder is without charge, there will be the loss in voltage

(712) $C\,(V_0' - V_1')$, (without charge).

When the rod is charged the loss in the same interval is,

(713) $C\,(V_0 - V_1)$, (with charge).

This change in the potential is caused by the passage of N ions, each with charge e, in the volume of air A per second, so that in volts with the factor 300, we have the equation, containing correction for the leakage by (695), for positive ions,

(714) $+\,e\,N_+\,A\,300 = C\,[(V_0 - V_1) - (V_0' - V_1')]$ in cubic meters.

(715) $n_+ = N \times 10^{-6} = C\,[(V_0 - V_1) - (V_0' - V_1')]\,\dfrac{2941}{300\,A}$ in cu. cms.

The passage of n_+ positive ions per cu. cm. has neutralized a certain amount of the charge of the opposite sign, so that by charging V_0, V_0' for one sign, the number of ions of the opposite sign is determined.

Example of the Computation

A 1 run = 0.185283 cubic meters. $C = 9.69$.

A_5 (560 seconds) = 0.926417 cubic meters. $\dfrac{C \times 2941}{300\,A} = 102.54$

(+) $(V_0 - V_1) = 9.0.$	$(V_0' - V_1') = 0.6.$
For positive charge,	$n_- = 84 \times 102.54 = 861.$
(−) $(V_0 - V_1) = 10.2$	$(V_0' - V_1') = 0.5.$
For negative charge,	$n_+ = 9.7 \times 102.54 = 995.$

The Formulas for the Velocity of the Ions

The velocity of motion of the positive and the negative ions respectively depends upon a special proposition, as follows:

(716) $\frac{dr}{dx} = -\frac{F}{G}u, \qquad G\,dr = -F\,u\,dx.$

(717) $F = -\frac{dV}{dr} = \frac{2Q}{r} = \frac{2}{r} \cdot \frac{V_0 - V_1}{2 \log_e \frac{r_1}{r_2}}$, by (678) and (682).

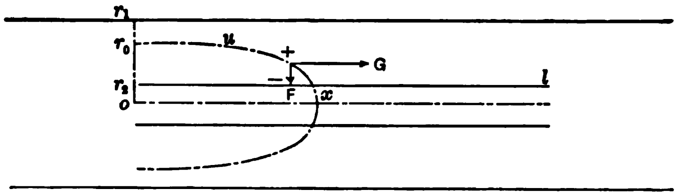


FIG. 56. The parabolic motion of the ions in a condenser of co-axial cylinders charged to V. u = the velocity of motion in cm/sec. F = the radial component along r. G = the axial component along x.

(718) $\frac{dr}{dx} = -\frac{u}{G} \cdot \frac{V_0 - V_1}{r \log_e \frac{r_1}{r_2}}$, for the unit charge. Integrate after transposing,

(719) $r^2 = -\frac{2u}{G} \cdot \frac{V_0 - V_1}{\log_e \frac{r_1}{r_2}} x + C.$ $C = r_0^2$ on ion entering the condenser field at r_0.

(720) $$x = (r_0^2 - r^2)\frac{G \log_e \frac{r_1}{r_2}}{2u(V_0 - V_1)}.$$

The maximum value of x is at a distance less than l from the origin, where the ion entering at r_0 falls finally upon the charged surface of the inner electrode.

(721) $$\frac{V_0 - V_1}{G} \gtreqless (r_1^2 - r_2^2)\frac{2 \log_e \frac{r_1}{r_2}}{4ul}, \quad \begin{bmatrix} > \text{falls inside } l \\ = \text{falls on end } l \\ < \text{falls outside } l \end{bmatrix}$$ Condition that all the ions fall upon electrode.

(722) $\frac{V_0 - V_1}{G} = (r_0^2 - r_2^2)\frac{2 \log_e \frac{r_1}{r_2}}{4\pi l}$, equation of condition.

The Ebert velocity apparatus consists of the primary condenser and electroscope, used for counting the number of ions, and in addition an auxiliary condenser in the same axial line, the inner electrode charged to a few volts ΔV, and the outer connected to earth.

Let $V_0 - V_1$ = the loss in voltage with no charge on the auxiliary.

$V_0' - V_1'$ = the loss when the auxiliary is charged to about $\Delta V = 30$ volts. As some ions fall on the auxiliary electrode, it follows that $(V_0 - V_1) > (V_0' - V_1')$

The amount of electricity drawn down upon the auxiliary electrode by its charge of opposite sign is,

$$(723)\quad (e\,n\,u)\,4\,\pi\,l\,(t_1 - t_0) = A\,\frac{(V_0 - V_1) - V_0' - V_1')}{\Delta V}.$$

We have, also, $e\,n = C\,(V_0 - V_1) = \dfrac{V_0 - V_1}{2\,\log_e \dfrac{r_1}{r_2}}$. Hence,

$$(724)\quad u = \frac{A}{t_1 - t_0}\cdot\frac{1}{4\,\pi\,l}\cdot\frac{1}{\Delta V}\cdot\frac{1}{M}\cdot 2\log\frac{r_1}{r_2}\cdot \frac{(V_0 - V_1) - (V_0' - V_1')}{V_0 - V_1}.$$

Example of the Velocity Computation

$$\frac{A}{t_1 - t_0} = \frac{0.85065 \times 10^6}{515} = 1652\,\frac{\text{cubic centimeters}}{\text{per second}}.$$

$$\frac{1}{4\,\pi\,l\,.\,\Delta V} = \frac{1}{4\,\pi \times 12 \times 27} = \frac{1}{4071}.$$

$$\frac{1}{M}\cdot 2\log\frac{r_1}{r_2} = 2.3026 \times 2 \times (\log 1.50 - \log 0.25) = 3.584.$$

$$\text{Constant} = \frac{1652 \times 3.584}{4071} = 1.45.$$

$$u = 1.45\,\frac{(V_0 - V_1) - (V_0' - V_1')}{V_0 - V_1},$$

$V_0 - V_1$ auxiliary not charged.
$V_0' - V_1'$ auxiliary charged.

$$(-)\ \text{Charge.}\quad u_+ = 1.45\,\frac{(190.7 - 177.1) - (184.9 - 177.6)}{190.7 - 177.1} =$$

$$1.45\,\frac{13.6 - 7.3}{13.6} = 0.67\ \text{cm/sec.}$$

$$(+)\ \text{Charge.}\quad u_- = 1.45\,\frac{(192.8 - 179.0) - (183.9 - 176.1)}{192.8 - 179.0} =$$

$$1.45\,\frac{13.8 - 7.8}{13.8} = 0.64\ \text{cm/sec.}$$

The Gerdien apparatus for the number and velocity of ions. This consists of a double condenser, fitted with a turbine for a measured quantity of air to be drawn through the tubes in the unit time. There are two electrometers, I with a variable capacity C', II with a constant capacity C. The outer cylinder has the radius $r_1 = 4.9$ cm., and the length $l = 65$ cm.; the auxiliary with variable capacity C' has the radius $r_2 = 0.5$ cm., and $l = 20.1$ cm.; the principal with constant capacity has the radius $r_2 = 1.0$ cm., and $l = 35$ cm. $C' = 16.7$, $C = 20.2$, in some instruments.

The number of ions. The capacities are both at a minimum, and the number of ions falling on both electrodes neutralizes a quantity of electricity of the *opposite sign*, expressed by the equation (725), where A = the number of cubic centimeters of air that passes in the time $(t_1 - t_0) = 10800000$ cm.^{3} for 80 revolutions.

(725) $e\,n\,A\,300 = C'\,(V_0' - V_1') + C\,(V_0 - V_1).$

(726) $n = [(C'\,(V_0' - V') \times C\,(V_0 - V_1)]\,0.908.$

The velocity of the ions per second. The capacity C' is increased to C_1', from 16.7 to 124, by raising the capacity cylinder 25 divisions. From (722),

$$(727)\quad \frac{V_0' - V_1'}{G} = (r_0^2 - r_2^2)\,\frac{2\,\log_e \frac{r_1}{r_2}}{4\,\pi\,l}.$$

The quantity of electricity on n-ions entering the field through the area $\pi\,(r_0^2 - r_1^2)$ falls on the electrode, so that,

$$(728)\quad \pi\,(r_0^2 - r_2^2) = 4\,\pi\,l.\,u.\,C\,V' = 4\,\pi\,l.\,u.\,\frac{V'}{2\,\log_e \frac{r_1}{r_2}}.$$

Hence, by substitutions, and for the interval $d\,t$,

$$(729)\quad e\,n\,.\,\pi\,(r_0^2 - r_2^2)\,G\,d\,t = e\,n\,.\,4\,\pi\,\lambda\,u\,\frac{V'}{2\,\log_e \frac{r_1}{r_2}}\,d\,t = - C_1'.d\,V'.$$

$$(730)\quad u = \frac{1}{e\,n\,(t_1 - t_0)}\left[\frac{1}{4\,\pi\,l}\cdot\frac{1}{M}\cdot 2\,\log\frac{r_1}{r_2}\,C_1'\cdot\frac{1}{M}\right]\log\frac{V_0'}{V'}.$$

$$(731)\quad u_+ = \frac{(e\,n\,u)_+}{e\,n_+}.\qquad u_- = \frac{(e\,n\,u)_-}{e\,n_-}.\qquad \text{Velocity in a field 1 volt/cm. E. S. U.}$$

$$(732)\quad \lambda_+ = (e\,n_+\,u_+)\,300.\quad \lambda_- = (e\,n_-\,u_-)\,300.\quad \lambda = \lambda_+ + \lambda_-.$$

Example of the Gerdien Computation

Compare Mollá, Tortosa Observatory

(−) Negative charge for the positive ions neutralized.

					I. $C^1 = 16.7$					II. $C = 20.2$			
	H	m	s	Index	L	R	S	V^1	log V^1	L	R	S	V
t_0	9	40	18	60	11.2	10.7	21.9	199.8	2.80060	12.0	15.6	27.6	209.4
t_1	10	6	33	140	9.8	9.8	19.1	174.0	2.24055	9.0	10.5	19.5	145.5
		26	15	80 A			$V^1_0 - V^1$	25.8	0.06005			$V_0 - V_1$	68.9
												$C\ (V_0 - V_1)$	1290.8
$t_1 - t_0$		1575			log. log V^1_0/V_1^1				8.77851			$C^1\ (V_0^1 - V_1^1)$	480.9
					[Constant]				8.28544				1721.7
log $(t_1 - t_0)$	8.19728				log. Const. log V_0^1/V_1^1				7.01895			Factor	0.908
												n_+	156.3
log $e\,n_+$	8.72544				log $en_+\,(t_1 - t_0)$				6.92272			log n_+	8.19396
log u_+	0.09123				log u_+				0.09123			log e	0.53148
log 300	2.47712				u_+				1.234			log $(t_1 - t_0)$	8.19728
log λ_+	6.29879	$\lambda_+ = 1.967 \times 10^{-4}$										log. $en_+\,(t_1 - t_0)$	6.92272

Read the scales for V_0' (auxiliary), and V_0 (principal), while C' is at minimum value 16.7. Raise the C' to $C_1' = 124$ by 25 divisions. Start the turbine and read the hour and index; after an interval for 80 A read the hour and index. Lower the capacity to minimum $C' = 16.7$ and read V_1' and V'. The formulas become by evaluation:

$$u = \frac{1}{e\,n} \cdot \frac{1}{t_1 - t_0}\, 0.0172 \log \frac{V_0'}{V_1'} \text{ (auxiliary).}$$

$$\lambda = e\,n\,u\,300.$$

$$4\,\pi\,\lambda = a.$$

$$\lambda = \lambda_+ + \lambda_- .$$

Example Continued.

(+) Positive charge for the negative ions neutralized.

	I. $C^1 = 16.7$									II. $C = 20.2$			
	H	m	s	Index	L	R	S	V'	log V'	L	R	S	V
t_0	10	11	30	65	10.0	10.7	20.7	188.4	2.27508	12.5	15.8	28.8	213.9
t_1		37	5	145	9.5	9.5	19.0	173.4	2.23905	9.9	11.7	21.6	162.0
		25	35	80 A			$V_0^1 - V_1^1$	15.0	0.03603		$V_0 - V_1$		51.9
$t_1 - t_0$			1535			log. log. V_0^1/V_1^1			8.55666		$C(V_0 - V_1)$		1048.4
						[Constant]			8.23544		$C^1(V_0^1 - V_1^1)$		250.5
log $(t_1 - t_0)$			318611			log Const. log. V_0^1/V_1^1			6.79210				1298.9
											Factor		0.908
log en_-			3.60299			log en_- $(t_1 - t_0)$			6.78910		n_-		1179
log u_-			0.00300			log u_-			0.00300		log u_-		3.07151
log 300			2.47712			u_-			1.007		log e		0.53148
											log $(t_1 - t_0)$		3.18611
log γ_-			6.08311	$\lambda_- = \lambda\ 1.211 \times 10^{-4}$		$\lambda = \lambda_+ + \lambda_- = 3.178 \times 10^{-4}$					log $e\,n_-$ $(t_1 - t_0)$		6.78910

The Cause of the Discrepancy in the Values of the Conductivity of the Atmosphere, as Determined by Two Methods

It will be seen from the collection of results in Table 68 that we have arrived at two very different results for the coefficient of atmospheric conductivity λ, as determined by the two methods that have been described. Section I is compiled in part from Mache and Schweidler's volume, and in part from Bigelow's observations of 1905, made during the U. S. Eclipse Expedition to Spain and Algeria, and of 1912 made at Cordoba, Argentina, with the resulting value of $\lambda = 5.575 \times 10^{-5}$ E. S. U. Section II is compiled from the same sources, with the result $\lambda = 20.49 \times 10^{-5}$, so that the ion counters give a value for λ which is 3.68 times greater than by the dissipation apparatus. The cause of this great discrepancy is not far to seek. The dissipation apparatus is generally worked under a cylindrical hood, which converts it into a condenser, through which the air current is not freely passing, with the effect that the free dissipation is transformed into a saturated or stagnant circulation of the ions. It is easy to show that this apparatus with the hood does not respond to the capacity formula upon which the computations are based. We have used cylinders of different dimensions, with the hood

TABLE 68

COMPARISON OF THE COMPUTED VALUES OF λ FROM THE DATA OBSERVED BY THE TWO METHODS

Section I. Elster and Geitel dissipation apparatus

Stations	a^1_+	a^1_-	q	a_+	a_-	λ_+	λ_-	λ
Lugano	1.87	1.87	1.00	3.12	3.12×10⁻⁴	2.48	2.48	4.96 ×10⁻⁵
Capri	4.22	4.44	1.05	7.04	7.40	5.59	5.87	11.46
Tromsö	3.02	3.38	1.11	5.03	5.63	3.99	4.47	8.46
Spitzbergen, land	3.76	5.86	1.55	6.27	9.77	4.98	7.75	12.73
Spitzbergen, sea	2.88	4.62	1.60	4.80	7.70	3.81	6.11	9.92
Juist	1.17	1.54	1.32	1.95	2.57	1.55	2.04	3.59
Wolfenbuttel	1.26	1.34	1.06	2.10	2.23	1.67	1.77	3.44
Misdroy	0.84	1.33	1.58	1.40	2.22	1.11	1.76	2.87
Ostsee	0.72	1.00	1.43	1.20	1.67	0.95	1.33	1.28
Potsdam	0.97	1.29	1.36	1.62	2.15	1.29	1.71	3.00
Kremsmünster	1.26	1.43	1.13	2.10	2.38	1.67	1.89	3.56
Triest	0.55	0.60	1.09	0.92	1.00	0.73	0.79	1.52
Karasjok	3.33	3.82	1.15	5.55	6.37	4.40	5.06	9.46
Daroca, 1905	1.36	1.39	1.02	2.27	2.32	1.80	1.84	3.64
Porta Cœli	1.45	1.55	1.07	2.42	2.58	1.92	2.05	3.97
Guelma	3.63	3.77	1.04	6.05	6.28	4.80	4.98	9.78
Bona	1.87	1.90	1.02	3.12	3.17	2.48	2.52	5.00
Cæsar, bulkhead	0.73	0.79	1.08	1.22	1.32	0.97	1.05	2.02
Cæsar, hatch	1.50	1.56	1.04	2.50	2.60	1.99	2.06	4.05
Cordoba, 1912	2.59	2.53	0.98	4.32	4.22	3.43	3.35	6.78
								5.575×10⁻⁵

and without the hood, leading to the following results: The cylinders are as indicated,

TABLE 68—(*Continued*)

Section II. Ebert's and Gerdien's Ion counters. $\lambda = (e\,n_+u_+ + en_-u_-)$ 300 E. S. U.

Stations	n_+	n_-	u_+	u_-	λ_+	λ_-	λ	$i = \lambda \frac{dv}{dh}$
Helgoland, düne....	382	206	.....	.....	.....	.....	....	....
Helgoland, oberland.	735	382	.....	.....	.....	.....	....	....
Swinemünde.......	823	647	.....	.....	.....	.....	....	....
Potsdam..........	1088	882	.....	.....	.....	.....	....	....
Atlantic Ocean.....	1147	794	.....	.....	.....	.....	....	....
Seewalchen........	1323	1117	1.02	1.25	13.76	14.24	28.00×10^{-8}	9.33×10^{-7}
Mattsee..........	1029	853	.....	.....	.....	.....	....	....
Säntis...........	1264	382	.....	.....	.....	.....	....	....
München..........	1558	1238	.....	.....	.....	.....	....	....
Bayern Alpental.....	1793	823	.....	.....	.....	.....	....	....
Golf von Lyon......	559	529	.....	.....	.....	.....	....	....
Mallorca..........	1176	1205	0.83	0.90	9.96	11.06	20.02	6.67
Barcelona.........	828	676	.....	.....	.....	.....	....	....
Karasjok..........	1117	970	.....	.....	.....	.....	....	....
Freiburg..........	1000	735	1.00	1.11	10.20	8.32	18.52	6.17
Stiller Ocean.......	588	588	.....	.....	.....	.....	....	....
Göttingen.........	....	....	1.32	1.40	.....	.....	....	....
Daroca, 1905.......	999	819	0.684	0.771	6.97	6.43	13.40	4.47
Guelma...........	2940	2960	.....	.....	.....	.....	....	....
Bona.............	1155	972	.....	.....	.....	.....	....	....
Cæsar, bulkhead....	621	626	.....	.....	.....	.....	....	....
Cæsar, hatch.......	1013	1166	.....	.....	.....	.....	....	....
Cæsar, outward.....	1275	1206	.....	.....	.....	.....	....	....
Cordoba, Sep. 12 to Oct. 10, 1912.....	1558	1542	0.701	0.613	10.77	10.30	21.07	7.02
Oct. 11–26.......	1733	1522	0.715	0.766	11.83	10.08	21.91	7.30
							20.49×10^{-8}	6.83×10^{-7}

The value of λ by the Ebert or Gerdien methods is more than ***three times greater*** than by the Elster and Geitel method.

(1). $r = 0.25$ cm. (2). $r = 2.50$ cm. (3). $r = 2.50$ cm.

$l = 5.00$ cm. $l = 10.00$ cm. $l = 20.00$ cm.

$K = 0.65$. $K_0 = 5.46$. $K_0 = 10.91$.

The capacity of the electroscope is 4.36 cm.

The capacity coefficients *without* hood become,

$$\frac{K+C}{K} = 7.71. \quad \frac{K+C}{K} = 1.80. \quad \frac{K+C}{K} = 1.40.$$

$$\frac{C}{K+C} = 0.87. \quad \frac{C}{K+C} = 0.44. \quad \frac{C}{K+C} = 0.29.$$

The capacity coefficients *with the hood* become:

$$K = 1.34. \quad K = 7.50. \quad K = 15.00.$$

$$\frac{K+C}{K} = 4.25. \quad \frac{K+C}{K} = 1.58. \quad \frac{K+C}{K} = 1.29.$$

$$\frac{C}{K+C} = 0.76. \quad \frac{C}{K+C} = 0.37. \quad \frac{C}{K+C} = 0.23.$$

Two sets of experiments have been carried out, one in 1905 and the other in 1912. In the Eclipse expedition the hood was used throughout the observations, but the dissipation bodies (1) and (2) were used in frequent interchanges. The following table shows the average fall in scale divisions during fifteen-minute intervals, with (2) the cylinder, and (1) the small rod, respectively:

TABLE 69

MEAN LOSS IN SCALE DIVISIONS IN FIFTEEN-MINUTE INTERVALS

Station	Daroca			Porta Cœli			U. S. S. *Cæsar* Bulkhead			U. S. S. *Cæsar* Hatch		
	No.	(+)	(−)	No.	+	(−)	No.	(+)	(−)	No.	(+)	(−)
Cylinder 10 x 5 cm.	657	−5.4	−5.4	354	−5.8	−6.3	147	−3.4	−3.6	126	−6.5	−6.7
Rod 3 x 0.5 cm.	139	−5.3	−5.5	80	−5.1	−5.8	32	−3.3	−3.4	25	−6.6	−6.6

The observations at Daroca and Porta Cœli were continued throughout the twenty-four hours. It is seen that the size of the dissipating body under the hood does not effectively control the rate of dissipation, and that the computed capacity coefficient does not properly enter into the computation.

A series of experiments was made at Cordoba, 1912, with the three dissipation bodies, in part with, and in part without, the hood. The mean values of the loss in scale divisions are proportional to the following data:

TABLE 70

MEAN LOSSES IN SCALE DIVISIONS WITHOUT AND WITH HOOD

Dissipation Body	Without Hood			With Hood		
	(+)	(−)	Mean	(+)	(−)	Mean
(1) Rod............	1.58	1.32	1.45	2.07	1.73	1.90
(2) Small cylinder...	4.82	3.98	4.40	1.63	2.07	1.85
(3) Large cylinder...	5.60	5.12	5.36	2.20	1.97	2.09
				1.98	1.92	1.95

It is seen that with the hood the size of the dissipation body is an indifferent matter, but that without the hood the size of the body and the rate of dissipation progress together.

A special observing shelter at Cordoba was constructed of canvas and netting, so that the dissipation observations could be made without using the hood, and the results for the series are recorded in Table 68. Referring to the capacity coefficients for the three bodies, it is seen that $\frac{K+C}{K}$ increases as K diminishes, the upper limit for $K = 0$ being ∞, and for $K = \infty$ being 1. We may, therefore, increase the coefficient by decreasing the size of the dissipation body. Since most of the experiments of Table 68 were made with the hood and for $\frac{K+C}{K} = 1.58$ (on the average), while, without the hood in Cordoba, $\frac{K+C}{K} =$ 1.80, it is apparent that the comparatively large value $\lambda = 6.78 \times 10^{-5}$ is due to this fact, at least in part. We can make a cylinder for a proper dissipation body which will give $5.575 \times 3.68 =$ 20.49. Its approximate size can be obtained by computation, but its actual and correct size is a matter for experiment. Such experiments were executed at La Quiaca, 1913, with the result that a dissipating cylinder 52 centimeters long by 1 centimeter in diameter will give nearly the same value of λ as the ion-counters. Similar results have been obtained in Potsdam. Such a long cylinder is very difficult to charge to a sufficiently high potential V and it is impractical. The Elster and Geitel apparatus without hood is valuable for giving relative values of $(a_+ \; a_-)$ $(\lambda_+ \; \lambda_-)$, and not their absolute values. These must be obtained by means of the Ebert or Gerdien ion-counters.

In *Terrestrial Magnetism*, Vol. XVIII, No. 4, Vol. XIX, No. 1, No. 2, No. 3, 1914, Mr. W. F. G. Swann discusses the theories of the several ionization apparatus, and indicates that several corrections must be applied in order to remove certain errors. These depend upon the following conditions:

1. The variation of the atmospheric potential gradient with the height above the surface to the axis of the horizontal cylinder. The change may amount to 20%.

2. The change in the shape of the electric stream lines on entering the cylinder, due the electrostatic attractions and repulsions, whereby the number of ions entering may vary as much as 20% to 30%.

3. The modification of the capacity of the inner condenser, which should contain the stub of the supporting rod outside the electroscope as well as the small inner charged cylinder. This may increase the "computed capacity" into the "measured capacity" by as much as 30% to 40%.

4. The value of the unit electric charge, as determined by several different lines of experiments, has been taken 3.4×10^{-10}, but the electrical constant of Table 95, $e = 4.774 \times 10^{-10}$, seems to require this increase in the computation of the coefficient of conductivity λ. It is evident that the entire subject is in need of further discussion.

The Atmospheric Electric Potential

The distribution of the electricity in the atmosphere consists normally of a heavy positive charge of several thousand volts at about 5,000 meters altitude, which diminishes toward the earth, with an increasing potential fall near the surface, the latter being charged with a negative potential. There are continual fluctuations among these potentials in diurnal, annual, and other periods, or in aperiodic variations as in thunderstorms and minor changes.

Let V = the potential at any height.

(733) Force $F = -\frac{dV}{dh} = 4\pi\sigma$, where σ is the surface density. (699)

(734) $-\frac{dF}{dh} = +\frac{d^2V}{dh^2} = -4\pi\rho$, where ρ is the volume density.

$$-\frac{dV}{dh} = -\frac{\text{volts}}{\text{meter}} = -\frac{\text{volts}}{100\text{ cm.}} = -\frac{\text{E. S. U.}}{300 \times 100\text{ cm.}}.$$

Surface density for the average potential fall of 100 volts/m.

(735) $\sigma = -\frac{100}{4\pi \times 300 \times 100} = -2.65 \times 10^{-4}$ E. S. U.

The potential gradient changes at about $-1/1000 = \frac{d^2V}{dh^2}$.

(736) $\rho = -\frac{d^2V}{dh^2} \cdot \frac{1}{4\pi} = -\frac{1}{12.56}\left(\frac{-1}{1000}\right)\frac{1}{300} = +2.7 \times 10^{-9}$ E. S. U.

The total surface charge of the earth is computed from σ with the radius of the earth $r = 6.37 \times 10^8$ cms., by (696),

(737) $Q_E = 4\pi r^2 . \sigma = 12.56 \times (6.37 \times 10^8)^2 . (-2.65 \times 10^{-4}) = -1.35 \times 10^{15}$.

Since 1 coulomb = 3×10^9 E. S. U.,

$Q_E = -4.5 \times 10^5$ E. S. U.

The electric pressure $U_l = U_t = \frac{F^2}{8\pi} = \frac{16\pi^2\sigma^2}{8\pi} = 2\pi\sigma^2$, by (672).

(738) $2\pi\sigma^2 = 4.43 \times 10^{-7}$ dynes.

The vertical electric current for $\lambda = 20.5 \times 10^{-5}$ E. S. U.

(739) $i = \lambda\frac{dV}{dh} = 20.5 \times 10^{-5}\frac{100\ \text{(volts)}}{300 \times 100} = 6.83 \times 10^{-7}$ E. S. U. by (700).

$i = 2.05 \times 10^{-15}$ amperes/cm.^{2}, since 1 ampere = 3×10^{-9} E. S. U.

The theory adopted in this work of the electric potential and its gradient observed in the lower atmosphere is that the incoming radiation ionizes the aqueous vapor in the strata within

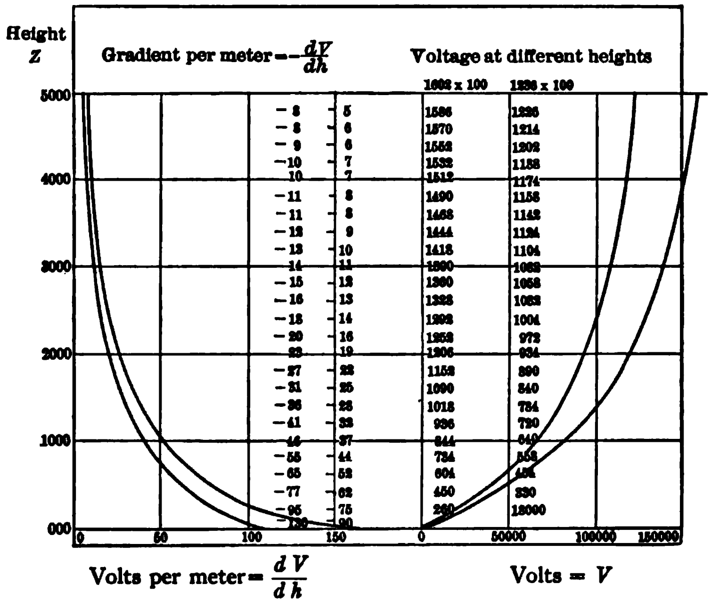


FIG. 57. The electric potential fall and the voltage at different heights.

a few thousand meters of the surface, so that the normal charge is 150,000 volts at 5,000 meters, 100,000 volts at 1,400 meters, and 0 volts at the surface, with negative induction in the earth itself. A study of the radiation data of Table 67, those of Table 63 for the distribution of the aqueous vapor, together with those of Fig. 57 for the electric potential, may lead to the function connecting radiation, ionization, aqueous vapor contents, electric currents, and diurnal magnetic deflecting vectors. This subject will require prolonged research in observation and analysis. Table 57 gives the distribution in heights for two cases of the voltage (160200, 123600), and the voltage gradient per 100 meters (−8 −5) at 5000 meters, with (−130 −90) at the surface where $V = 0$ in both cases.

CHAPTER VI

Terrestrial and Solar Relations

The Five Types of the Diurnal Convection in the Earth's Atmosphere

THE further analysis of the problems of electric and magnetic variations depends upon the determination of the types of the diurnal convections in the earth's atmosphere. There are five of these types, distinct from one another: (1) In the Arctic zone; (2) in the North Temperate zone, Lat. + 66° to + 30°; (3) in the Tropic zone, Lat. + 30° to − 30°; (4) in the South Temperate zone, Lat. − 30° to − 66°, and (5) in the Antarctic zone. What is needed is a complete determination of the diurnal deflecting wind vectors, for each hour of the day and night, and on several planes from the surface to 3,000 meters, as 000, 200, 400, 600, 800, 1,000, 1,500, 2,000, 2,500, 3,000 meters. These can be obtained by kites or captive balloons, but the labor will be not inconsiderable. Unfortunately the available material is very meager, and it is almost wholly lacking during the night. Without the night observations those made during the day, 6 A.M. to 6 P.M., are of quite subordinate value. At present we have data made during the day and night only at the Blue Hill Observatory, 1897–1902, analyzed in my papers, "Studies on the Diurnal Periods in the Lower Strata of the Atmosphere," *Monthly Weather Review*, February to August, 1905; at several Mountain Observatories, J. Hann, K. Ak. Wiss., Wien, Bd. CXI, Abth. IIa, December, 1902, and April, 1903; there are several stations on the surface which can be utilized for provisional discussions, Wien, Mauritius, Batavia, Cordoba, Chacarita, Laurie Island in the South Orkneys. The circulation in the North Temperate zone can be quite accurately constructed, while that in each of the other zones can only be provisionally inferred. The available data have been thoroughly recomputed with the results collected in Table 71, where s = the velocity in meters per second, and β = the azimuth angle from $S = 0°$

TABLE 71

HIGH-LEVEL STATIONS IN THE NORTHERN HEMISPHERE

STATION ELEVATION	PIKE'S PEAK 4308 m.		SONNBLICK 3106 m.		SÄNTIS GIPFEL 2510 m.				SÄNTIS 2500 m.				OBIR 2140 m.	
Season	Summer		Winter		Summer		Winter		Summer		Winter		Winter	
Hours	s	β	s	β	s	β	s	β	s	β	s	β	s	β
0 A.M.	0.84	100°	0.28	98°	0.75	105°	0.31	234°	0.61	117°	0.50	195°	0.22	1°
2	1.34	83	0.39	102	0.75	105	0.17	211	0.67	105	0.33	124	0.25	35
4	1.46	71	0.44	44	0.47	80	0.14	310	0.47	57	0.64	98	0.33	47
6	1.05	57	0.67	357	0.28	6	0.33	325	0.56	347	0.56	29	0.28	28
8	0.43	12	0.44	331	0.39	288	0.25	338	0.47	292	0.83	41	0.19	307
10	0.66	279	0.44	206	0.69	246	0.25	275	0.97	250	0.83	338	0.42	253
0 P.M.	1.03	262	1.08	182	0.86	230	0.67	112	1.03	222	0.47	186	0.56	226
2	1.09	256	1.08	179	0.69	233	0.72	110	0.64	201	0.69	180	0.50	197
4	0.95	253	0.36	190	0.33	282	0.33	96	0.14	90	0.33	334	0.36	155
6	0.74	247	0.53	337	0.50	349	0.22	314	0.58	28	0.61	275	0.28	105
8	0.49	47	0.78	346	0.53	24	0.53	287	0.58	30	0.89	315	0.22	47
10	0.36	153	0.42	0	0.53	76	0.47	263	0.36	80	0.42	276	0.25	3

TABLE 71—(*Continued*)

LOW-LEVEL STATIONS IN THE NORTHERN HEMISPHERE AND THE TROPICS

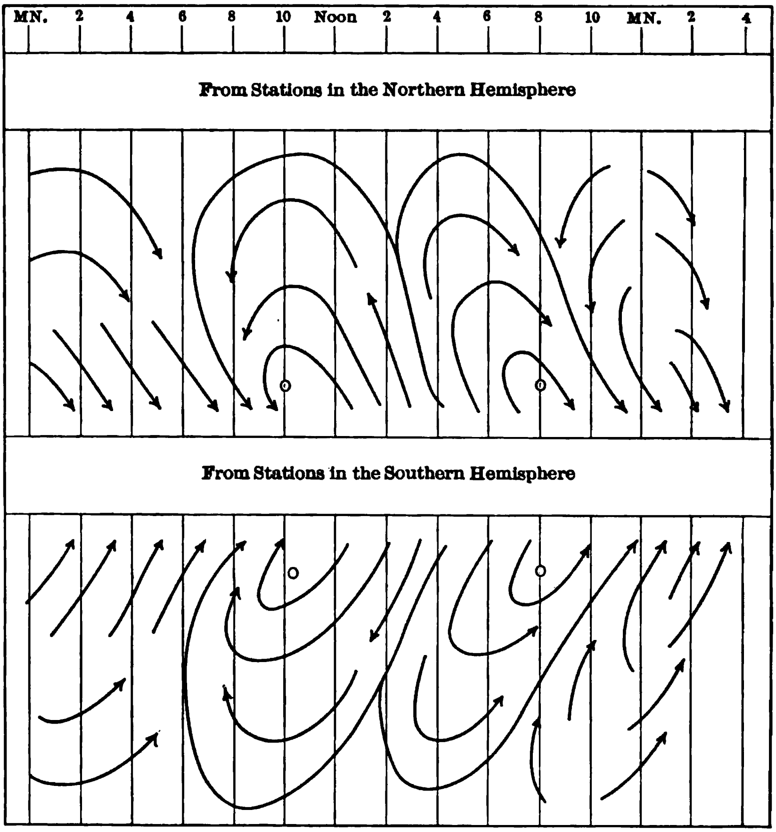
STATION ELEVATION	VIENNA 26 m.				BATAVIA 8 m.				MAURITIUS 15 m.			
Season	Summer		Winter		Dec.–Feb.		June–Aug.		Dec.–Feb.		June–Aug.	
Hours	*s*	β	*s*	β	*s*	β	*s*	β	*s*	β	*s*	β
0 A.M.	0.69	58°	0.47	62°	0.49	58°	0.68	305°	1.00	100°	0.70	70°
2	0.75	56	0.56	61	0.48	60	0.69	306	1.30	97	0.80	82
4	0.78	68	0.42	59	0.49	60	0.68	305	1.30	98	0.80	82
6	0.64	61	0.33	57	0.49	60	0.70	303	1.00	119	0.75	90
8	0.33	30	0.22	46	0.22	45	0.58	311	1.10	241	0.40	240
10	0.39	303	0.17	303	0.45	240	0.65	34	1.80	312	1.60	272
0 P.M.	0.78	265	0.36	257	1.33	200	0.85	100	2.40	326	1.30	288
2	0.83	243	0.58	242	1.58	193	1.77	159	1.60	332	1.00	287
4	0.94	224	0.64	232	0.82	192	1.92	158	1.30	304	0.90	280
6	0.89	217	0.47	223	0.26	58	0.16	232	0.20	10	0.40	63
8	0.25	186	0.14	186	0.41	54	0.65	303	0.90	101	0.50	68
10	0.44	65	0.25	72	0.50	59	0.69	304	1.00	102	0.70	65

TABLE 71—(*Continued*)

LOW-LEVEL STATIONS IN THE SOUTHERN HEMISPHERE

STATION ELEVATION	CORDOBA 437 m.								CHACARITA 25 m.				S. ORKNEYS 25 m.	
Season	Dec.–Feb.		Mch.–May		June–Aug.		Sep.–Nov.		Dec.–Feb.		Mch.–May.		Dec.–Feb.	
Hours	s	β	s	β	s	β	s	β	s	β	s	β	s	β
0 A.M	1.50	133°	0.97	127°	0.94	115°	1.61	125°	1.39	201°	0.69	198°	0.52	70°
2	2.22	134	1.39	126	1.06	111	2.03	128	0.53	200	0.25	110	0.52	51
4	2.61	134	1.50	129	1.44	121	2.39	129	0.58	59	0.53	71	0.51	30
6	2.75	129	1.97	131	2.03	132	2.42	129	1.78	49	0.67	70	0.51	5
8	1.08	135	1.75	135	2.17	136	0.97	128	2.53	37	1.25	42	0.52	343
10	2.33	340	1.08	262	0.50	252	1.94	333	2.53	30	1.31	11	0.53	262
0 P.M.	2.64	322	2.58	335	2.78	314	2.89	340	2.00	17	1.28	359	0.54	255
2	3.11	318	2.89	336	3.56	315	3.28	338	1.28	346	1.03	346	0.54	245
4	3.03	311	3.00	322	3.36	305	3.08	320	1.64	250	0.50	260	0.54	42
6	2.50	301	1.81	313	1.75	299	2.19	304	2.53	211	1.42	201	0.53	35
8	1.19	3	0.86	39	0.72	44	0.83	7	2.64	197	1.67	198	0.53	50
10	1.14	135	0.75	131	0.89	128	1.25	129	2.06	197	1.36	202	0.52	60

through E = 90°, N = 180°, W = 270°. The vector direction is that toward which the stream is moving. Having the mean observed hourly vectors, D, and the mean 24-hourly resultant R, these deflecting vectors D are such that $R + D = O$, constructed as true vectors of velocity and direction.



FIGS. 58 and 59. The probable types of the mean diurnal circulation.

The data of Table 71 were plotted in diagrams and the apparent circulation for the North Temperate zone was constructed, as in Fig. 58. By analogy, from the surface data of Cordoba and Chacarita, the circulation is constructed for the

South Temperate zone, Fig. 59. From Mauritius and Batavia, we have that from the Tropics; and those for the Arctic and the Antarctic zones are made from other data to be mentioned later. In the two Temperate zones the circulation is oppositely directed, in general, with turning-points at 10 A.M. and 8 P.M., with divergence at 2 to 4 P.M., and convergence at 6 to 8 A.M., and 8 to 12 P.M. The air is rising in the afternoon and falling in two streams during the night, as 10 to 12 P.M. and 2 to 8 A.M. This is the circulation which is caused by the diurnal heating of the earth's surface and the lower strata of the atmosphere with rising air during the daytime, and with cooling and descending air during the night. This circulation is limited to 3,000 meters from the surface, and it is not vigorous above 2,000 meters. The local conditions of mountain stations introduce many minor modifications, and such stations are never fully equivalent to ideal free-air conditions. The diagrams of Figs. 58, 59, contain the horizontal component chiefly, but the vertical component can be approximately inferred from the general stream lines. Similarly, we have the horizontal and the vertical circulations in the several zones, as may be seen in Fig. 64 in connection with the magnetic vector systems which depend upon them. The vectors of Fig. 64 are the ends of the stream lines at the surface, as determined by the data of Table 71. These data were actually applied to globe models, and from them the adopted circulation was derived.

The Diurnal Variations of the Meteorological, Electrical, and Magnetic Elements

The effect of the diurnal circulation on the several meteorological elements is very complex, and especially so in view of the incessant interchanges between the diurnal and the semi-diurnal periods in the lower strata. Table 72 summarizes some examples of this interconversion, which can be profitably studied by transferring the data to suitable diagrams.

1. The temperature data for the lower strata, 000 to 2,500 meters, are taken from Table 24; that for *B* is from the same

table, by the conversion from *P* to *B*. The temperature has a simple diurnal wave at the surface, as heretofore explained, but a semidiurnal wave above 500 meters, diminishing to extinction at about 3,000 meters. These results conform to the Blue Hill direct observations of temperature in the free air.

TABLE 72

EXAMPLES OF THE TRANSITION FROM THE SEMIDIURNAL PERIODS TO THE DIURNAL PERIOD OF DIFFERENT ELEMENTS.

1. Temperature *T*						2. Barometric Pressure *B*					
Meters	Cordoba					Meters	Cordoba				
	000	500	1000	1500	2500		000	500	1000	1500	2500
0 A.M.	292.8	295.0	291.0	287.7	280.9	0 A.M.	760.85	717.64	676.95	638.18	565.99
2	291.7	295.4	291.9	288.6	281.0	2	760.18	717.27	676.72	638.10	565.98
4	291.1	295.3	291.6	288.0	281.0	4	760.40	717.42	676.85	638.18	565.98
6	291.0	294.9	290.6	287.5	280.8	6	760.80	717.84	677.16	638.29	565.99
8	293.2	294.0	290.0	287.1	280.8	8	760.93	718.22	677.30	638.33	566.00
10	297.7	293.5	289.5	287.0	280.7	10	760.95	718.28	677.34	638.34	566.01
0 P.M.	300.2	295.0	290.7	287.7	280.8	0 P.M.	760.30	717.80	677.06	638.28	565.99
2	301.8	298.0	292.6	288.8	280.9	2	759.08	717.05	676.71	638.13	565.97
4	300.7	298.6	293.8	289.7	281.2	4	758.56	716.72	676.43	637.93	565.95
6	299.0	297.7	293.5	290.0	281.5	6	758.62	716.49	676.25	637.79	565.94
8	296.4	296.1	292.2	288.9	281.2	8	759.70	717.15	676.60	638.02	565.97
10	294.3	294.7	290.7	287.1	280.8	10	760.58	717.91	677.21	638.27	566.00

3. Vapor Pressure e_d						4. Vapor Pressure e_d					
Tower	Salton Sea, Tower No. 1					Tower	Salton Sea, Tower No. 4				
Pans Feet	(1) 00	(2) 10	(3) 20	(4) 30	(5) 40	Pans Feet	(1) 2	(2) 10	(3) 20	(4) 30	(5) 40
0 A.M.	11.7	11.7	11.6	11.8	12.9	0 A.M.	16.0	14.8	14.8	14.2	14.3
2	10.7	11.0	11.0	11.0	12.7	2	14.8	13.8	14.1	13.8	13.5
4	10.0	10.2	10.3	10.6	12.5	4	13.7	12.8	13.2	12.0	12.7
6	9.8	10.2	10.1	10.4	12.3	6	12.6	12.7	12.9	11.6	12.5
8	12.4	13.2	12.2	12.3	13.3	8	14.6	14.6	14.0	13.7	14.1
10	14.3	14.7	13.6	13.7	13.9	10	16.9	16.5	15.8	15.0	15.4
0 P.M.	12.1	12.1	10.6	11.2	12.0	0 P.M.	18.0	17.7	16.1	13.8	14.0
2	9.7	9.6	8.2	8.5	9.5	2	18.6	17.3	15.5	12.0	12.0
4	9.3	9.3	8.4	8.2	8.9	4	18.8	16.4	14.6	11.8	11.3
6	9.9	9.7	9.8	8.7	8.6	6	18.3	15.1	14.2	11.8	11.2
8	11.0	11.3	11.0	10.3	10.8	8	17.8	15.0	14.8	13.3	13.3
10	12.1	12.3	12.0	12.1	13.3	10	17.4	15.0	15.8	15.4	15.3

TABLE 72—(*Continued*)

5. Vapor Pressure e_d					6. Electric Potential Fall.						
Meters	Blue Hill (Summer)				Hours	Kew		Kremsmünster		Greenwich	
	000	200	400	1000		Sum.	Wint.	Sum.	Wint.	Sum.	Wint.
0 A.M.	11.07	12.15	8.78	7.53	0 A.M.	+ 3	− 8	−12	−33	+ 6	− 1
2	10.91	11.40	8.41	6.80	2	−14	−38	−19	−51	+ 1	− 9
4	10.75	11.16	8.08	6.44	4	−20	−58	−24	−54	− 3	−13
6	10.88	10.74	8.15	6.44	6	00	−36	+ 2	−23	+ 1	−13
8	11.25	10.75	8.45	6.67	8	+23	+ 8	+16	+ 4	+ 1	− 6
10	11.14	11.11	8.92	7.47	10	+14	+34	+13	+22	+ 5	+ 5
0 P.M.	10.67	11.62	9.48	8.19	0 P.M.	−10	+10	+ 7	+19	− 3	+ 4
2	10.50	11.53	10.08	8.84	2	−24	− 6	+ 4	+29	− 9	+ 3
4	10.46	11.12	10.27	9.02	4	−21	+ 3	+ 5	+23	− 3	+ 6
6	10.74	10.85	9.72	8.81	6	− 2	+25	+ 4	+35	− 3	+ 8
8	11.15	10.79	9.15	8.31	8	+24	+29	+11	+32	+ 1	+ 7
10	11.17	11.38	8.87	7.81	10	+27	+21	− 7	+ 3	+ 8	+ 7

2. The pressure B has semidiurnal waves from the surface upward, diminishing to extinction on the 3,000-meter level. The morning crest of maximum is smaller than that of the afternoon. There is not the least evidence that the semidiurnal pressure waves embrace an oscillation of the entire atmosphere, as Kelvin's theory of the forced oscillations demands, and therefore several discussions and other inferences depending on that theory are really without proper foundations.

3. The vapor pressure is subject to this interchange of periods. At Salton Sea, Southern California, Tower No. 1 was located in the desert, 1,500 feet from the water, and the semidiurnal period is clearly defined at every stage from the surface to 40 feet. Tower No. 4 was located in the sea, at one mile from the shore, and it was observed, as in section 4, that the diurnal vapor pressure e_d near the water converts itself into a semidiurnal wave within 40 feet of the water. At Tower No. 1 the diurnal convection could not obtain vapor from the surface sufficient to fill up the diurnal wave, while at the water this deficiency did not exist near the surface, in consequence of the rapid evaporation.

Section 5, for the vapor pressure in the free air at considerable heights above the surface of Blue Hill, shows a recombination of the semidiurnal waves at the surface into diurnal waves at less than 1,000 meters above the surface. Here the diurnal convection carries the vapor upward to cooler strata and there concentrates it into a single maximum at about 4 P.M. The details of the physical conditions of these periodic interchanges must be left for more minute researches into the prevailing forces that are at work.

Section 6 gives some examples of the well-known change from the semidiurnal waves of the electric potential fall, prevailing generally in the summer where there is vigorous convection in the lower strata, into the diurnal or approximately diurnal wave which is characteristic of the winter months. Here the positive ions in the atmosphere appear to move up and down, relatively to the surface, upward in the convection of the afternoon, thereby diminishing the potential gradient, and downward at 8 A.M., and 8 P.M., thereby increasing the potential gradients.

Table 73 contains a collection of the coefficient of dissipation a', in percentage per minute, from observations made throughout the 24 hours at Daroca, Porta Cœli, Guelma, Stations of the U. S. Eclipse Expedition, 1905, also at Bona, and on the U. S. S. *Cæsar* during the voyage across the Atlantic Ocean.

Daroca is on the Aragon plateau; Porta Cœli is near Valencia; Guelma is in the interior of Algeria and Bona is the port. The observations on the U. S. S. *Cæsar* were made during the voyage from Gibraltar to the United States: (1) on the forward hatch, and (2) under the shelter of a large iron bulkhead. The values of a'_+ and a'_- at Guelma have been multiplied by the factor ½, and those at U. S. S. *Cæsar*, bulkhead, by the factor 2, in order to reduce them to the scale of the other series in taking the general means. These data are plotted in Fig. 60 in percentage per minute a'.

Fig. 60 shows that the diurnal variation of the dissipation coefficient in percentage per minute has a maximum at about 3 P.M., corresponding with the vertical convection, and secondary maxima at 10 P.M., and 2 to 4 A.M., corresponding with the two descending branches. The minor crests occur at 1 A.M., 4 A.M., 8 A.M., 0 P.M., 3 P.M., 5 P.M., and 9 P.M. Each of them corresponds

TABLE 73

THE COEFFICIENT OF ELECTRIC DISSIPATION, *a*. U. S. ECLIPSE EXPEDITION TO SPAIN AND ALGERIA, 1905.

(+) The positive coefficient in % per min. (−) The negative coefficient in % per min.

Stations	Daroca	Porta Cœli	Guelma	Bona	*Cæsar* Blkd.	*Cæsar* Hatch	Means	Daroca	Porta Cœli	Guelma	Bona	*Cæsar* Blkd.	*Cæsar* Hatch	Means
0 A.M.	1.24	1.80	2.15	....	....	....	1.76	1.33	1.81	1.97	....	....	....	1.70
1	1.45	1.76	1.96	....	....	....	1.72	1.42	1.71	2.20	....	....	....	1.78
2	1.13	1.56	1.91	....	....	....	1.53	1.11	1.69	2.04	....	....	....	1.61
3	1.11	1.86	1.80	....	....	....	1.59	1.20	2.03	1.80	....	....	....	1.68
4	1.34	1.74	1.99	....	....	....	1.69	1.27	1.98	2.11	....	....	....	1.79
5	1.30	1.64	1.77	....	....	....	1.57	1.29	1.79	2.03	....	....	....	1.70
6	1.28	1.58	1.57	1.61	1.06	1.22	1.39	1.37	1.75	1.61	1.58	1.26	1.40	1.49
7	1.09	1.56	1.51	1.74	1.36	1.38	1.44	1.10	1.54	1.61	1.84	1.86	1.59	1.59
8	1.16	1.65	1.59	1.79	1.42	1.58	1.53	1.23	1.89	1.54	1.53	1.54	1.74	1.58
9	1.33	1.55	1.50	1.61	1.36	1.41	1.46	1.41	1.76	1.50	1.54	1.40	1.40	1.50
10	1.39	1.70	1.48	1.63	1.66	1.59	1.58	1.41	1.98	1.53	1.80	1.70	1.69	1.69
11	1.57	1.59	1.54	1.90	1.46	1.63	1.62	1.55	1.78	1.61	1.95	1.64	1.62	1.69
0 P.M.	1.48	1.73	1.59	1.95	1.84	1.58	1.70	1.49	1.98	1.71	1.96	1.78	1.84	1.79
1	1.58	1.63	1.78	1.93	1.44	1.52	1.65	1.63	1.74	2.01	2.05	1.76	1.58	1.80
2	1.53	1.67	1.97	1.87	1.26	1.49	1.63	1.48	1.87	2.06	1.97	1.16	1.51	1.84
3	1.48	1.47	2.14	2.09	1.72	1.56	1.74	1.47	1.66	2.03	2.31	1.76	1.57	1.79
4	1.61	1.69	2.10	1.78	1.40	1.53	1.69	1.57	1.83	2.09	2.04	1.54	1.59	1.78
5	1.56	1.72	1.68	2.14	1.88	1.71	1.78	1.65	1.80	1.99	2.27	1.80	1.52	1.84
6	1.41	1.58	1.83	....	....	1.34	1.53	1.45	1.84	1.75	....	....	....	1.68
7	1.35	1.51	1.65	....	....	....	1.50	1.55	1.83	1.86	....	....	....	1.75
8	1.40	1.82	1.68	....	....	....	1.63	1.46	1.85	1.84	....	....	....	1.75
9	1.39	1.97	2.15	....	....	....	1.84	1.41	1.95	2.17	....	....	....	1.84
10	1.20	1.65	2.13	....	....	....	1.63	1.23	1.91	2.06	....	....	....	1.73
11	1.22	1.74	2.21	....	....	....	1.76	1.19	1.52	2.20	....	....	....	1.64

with some feature of the diurnal circulation, as it transports the ions from one level to another in this complex system of local

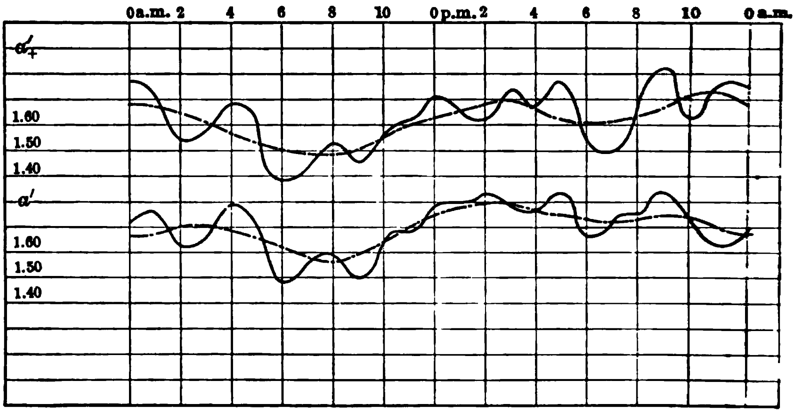


FIG. 60. Mean diurnal variation of the coefficient of electrical dissipation in percentage per minute.

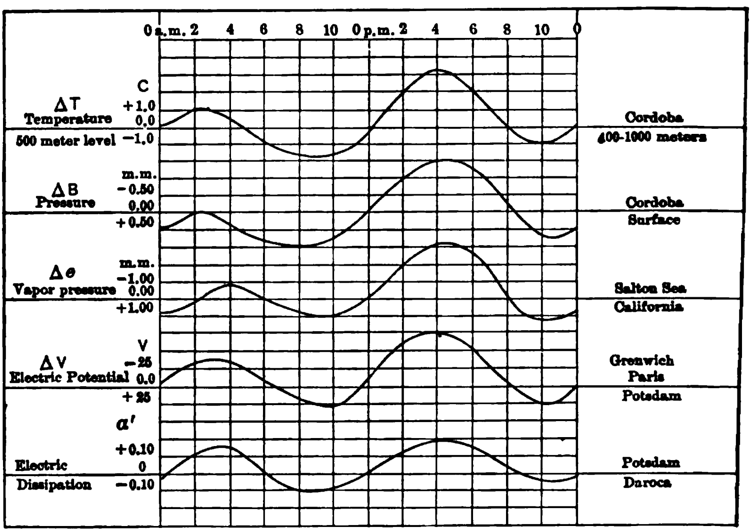


FIG. 61. Summary of the various semidiurnal waves.

circulations. These data should be much more extensively studied.

It is generally found that the coefficient of dissipation varies as follows: (1) Greater in clear air and less in cloudy, dusty air; (2) Greater with increase of the wind velocity; (3) Greater with increase of the temperature; (4) Greater with the higher vapor pressure. We may finally compare the diurnal curves of the several elements with the temperature waves in the strata 400–1,500 meters above the surface.

The distribution of the evaporation in the soil from the surface to 100 c.m. has been carefully worked out at Cordoba, with the result that the evaporation from the water table in the soil takes place in a diurnal curve, exactly agreeing with that of the vapor pressure. It is, therefore, thought that Δe, ΔV, a' of Fig. 61 should be inverted and referred to subsurface evaporation of ground water.

The Diurnal Variations of the Terrestrial Magnetic Field

There is another effect of the diurnal circulation in the earth's atmosphere, first, in the generation of electric currents, and

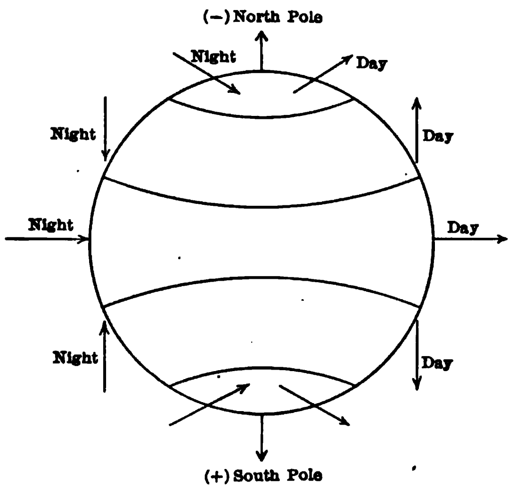


FIG. 62. Scheme of the diurnal circulation in zones.

secondarily, in the induction of the diurnal magnetic deflecting vectors that cause the variations of the normal magnetic field.

In Bulletin No. 21, U. S. W. B., 1898, were published the results of a computation on the observed elements H. D. V. at 30 stations.

TABLE 74

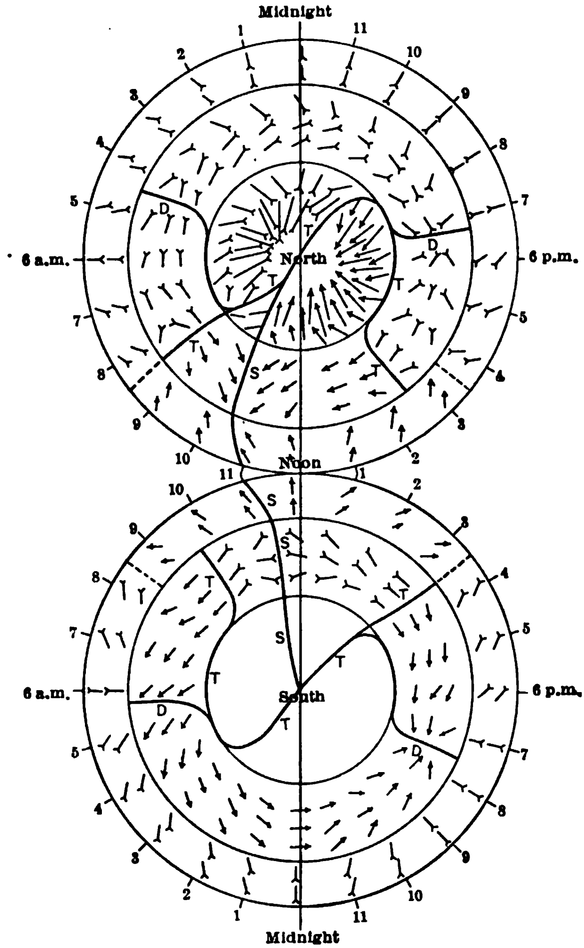
THE MEAN MAGNETIC DEFLECTING VECTORS IN FOUR ZONES

Hours of the Observations	Arctic Zone			North Temperate Zone			Tropic Zone			South Temperate Zone		
	Mag. Lat. 78° to 62° Stations 7			Mag. Lat. 61° to 28° Stations 13			Mag. Lat. + 10° to − 15° Stations 5			Mag. Lat. − 30° to − 55° Stations 5		
	s	α	β	s	α	β	s	α	β	s	α	β
Midnight	60	−36°	345°	15	−30°	111°	20	−33°	5°	19	+27°	259°
1 A.M.	63	−44	355	14	−35	109	19	−32	16	19	+31	250
2	69	−43	5	14	−32	102	20	−36	7	17	+35	251
3	74	−44	16	14	−33	108	20	−42	6	18	+36	243
4	75	−42	25	15	−35	112	18	−34	10	20	+36	226
5	77	−42	30	17	−33	110	17	−37	6	21	+33	223
6	78	−40	32	20	−31	112	19	−36	4	24	+31	222
7	76	−40	36	22	− 6	107	21	−37	339	26	+24	235
8	65	−37	45	25	+ 3	99	24	−30	297	28	+28	248
9	54	−18	68	26	+24	66	26	+23	228	28	+33	256
10	39	+31	117	27	+37	49	35	+25	210	26	−27	296
11	47	+44	195	25	+38	312	43	+22	204	25	−37	327
Noon	56	+43	200	33	+35	287	43	+30	193	28	−41	47
1 P.M.	64	+42	204	32	+26	277	40	+31	163	32	−36	53
2	73	+41	206	29	+23	268	34	+27	156	30	−35	72
3	83	+39	206	25	−24	263	17	+27	121	30	+30	82
4	89	+39	206	22	−39	259	16	−31	40	28	+30	85
5	87	+36	209	19	−45	260	16	−25	18	24	+39	84
6	78	+37	209	18	−54	234	19	−22	12	20	+40	84
7	62	+34	212	17	−44	183	21	−30	3	17	+45	78
8	54	+32	219	16	−40	183	23	−30	2	17	+46	108
9	51	+11	256	14	−39	103	23	−30	358	17	+47	283
10	50	− 6	279	15	−36	96	23	−28	2	18	+41	269
11	51	−37	336	13	−33	105	23	−28	6	18	+36	264

s = the vector in units of the 5th decimal (C. G. S.), 0.00001 dyne.
α = the vertical angle, positive above the horizon
β = the azimuth angle, from the South through E. N. W.

(740) $s = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}}$, $\sigma = (dx^2 + dy^2)^{\frac{1}{2}}$, $\tan \alpha = \frac{dz}{\sigma}$, $\tan \beta = \frac{dy}{dx}$.

Table 74 contains a condensed summary of these vectors, s, α, β, in the four principal zones (Fig. 62), that for the Antarctic being omitted for lack of observations. Fig. 63 contains a diagram of the vectors, which illustrates the system to some extent, though they can be properly studied and appreciated only by



Arrows with ⟶ for angles above the horizon, + a.
Arrows with ≻— for angles below the horizon, − a.

FIG. 63. Scheme of the directions of the deflecting forces causing the diurnal variations of the magnetic field in five principal zones. (Figure on page 90, Bulletin No. 21, U. S. W. B., 1898.)

reference to the original 30-inch globe model. The reader is directed to Bulletin No. 21 for further discussion of these data.

This magnetic system has constituted a difficult problem for solution, as it is necessary to have a simple, world-wide cause capable of producing these diverse effects.

The most prominent fact is the inversion of vectors as between the two hemispheres, and it is easy to show that the diurnal convection is oppositely directed in reference to the normal magnetic field, positive in the southern hemisphere and negative in the northern hemisphere.

In the Tropic zone the air rises nearly vertically by day and falls by night; in the Temperate zones it flows toward the poles by day, and toward the equator by night, being oppositely directed in each hemisphere relative to the positive direction of the magnetic field; in the Arctic and Antarctic zones the movement is upward by day toward the sun and downward at night. These five zones of circulation are marked off from each other by the high-pressure belts in latitudes + 30° and − 30° and by the low-pressure belts in latitudes + 66° and − 66°. The zones of circulation agree with the zones of magnetic vectors as defined in 1892.

Fig. 64 contains a scheme of the circulation vectors (black), and the magnetic vectors (dotted), as derived from the two sources indicated. There is remarkable agreement so far as the observational data extend, and the corresponding portions of the circulation adopted by natural inference agree with the parts that are known. It is generally true, (1) that the circulation vectors and the magnetic vectors are at right angles to each other, and (2) that the turning points in both systems coincide in all parts of the five zones. The conclusion is almost imperative that the circulating vectors, through the generated ions in streams, induce the observed magnetic deflecting vectors. While there is much to be done by observations fully to verify this theory, it is clear that the main features of both the systems are in remarkable conformity to the known facts of the observations. The horizontal and vertical components of the two sets of vectors in Fig. 64 should be united in one set of spacial vectors, in order that this system may be properly comprehended. The evidence is very strong that the magnetic variations depend

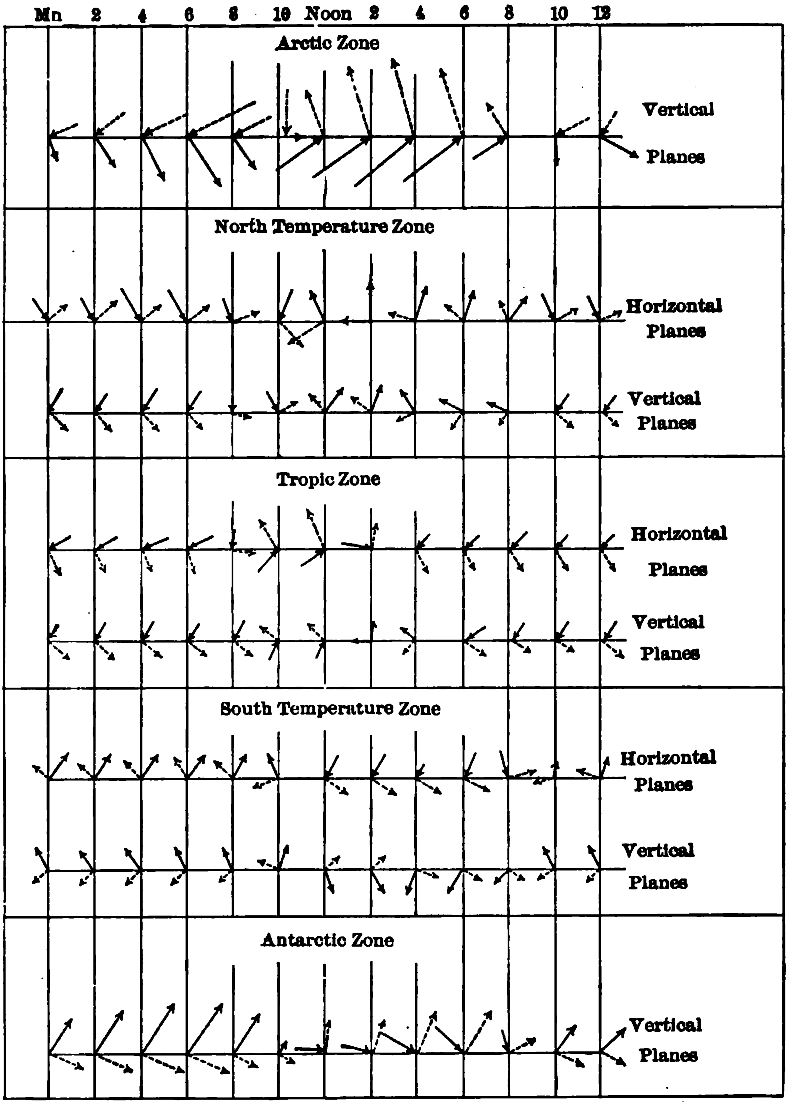


FIG. 64. The probable types of the diurnal wind vectors as generators of the diurnal magnetic deflecting vectors. Full-lined vectors = the electrical currents in the streams of the diurnal convection secondary vectors. Dotted vectors = the induced magnetic deflecting vectors as computed from the observations and given in the U. S. Weather Bureau Bulletin No. 21, page 87, 1897.

upon ionization in the *lower strata*, and not upon any system of ionization currents in the *upper strata*, as has been claimed.

Besides the vector directions of the magnetic forces we may approximately obtain the forces by the formula,

(741) $\Delta H = 4 \pi e n_+ u_+$,

where $e = 3.4 \times 10^{-10}$, n_+ = the number of positive ions per cubic centimeter, and u_+ the velocity of the circulation in centimeters per second. The available data for the level 400 meters are probably approximately as follows, in the South Temperate zone:

TABLE 75

THE DEFLECTING MAGNETIC VECTORS AS COMPUTED AND OBSERVED

$\Delta H = 4 \pi e n u$ in C. G. S. units.

Formula	A.M.					
	0	2	4	6	8	10
4π	12.56	......	......	......	......	......
e	3.4×10^{-10}	......	......	......	......	......
n_+	1045	1226	1374	1284	1245	1223
u_+	35	38	40	36	34	46
ΔH	.00016	.00020	.00023	.00028	.00018	.00024
Observed	.00019	.00017	.00020	.00024	.00028	.00026

Formula	P.M.					
	0	2	4	6	8	10
4π	......	......	......	......	......	......
e	......	......	......	......	......	......
n_+	1195	1232	1287	1225	1242	1274
u_+	75	70	55	50	45	38
ΔH	.00038	.00037	.00030	.00026	.00024	.00021
Observed	.00028	.00030	.00028	.00020	.00017	.00018

The number of ions per cubic centimeter was obtained at the surface at Daroca and Guelma, 1905, and from the other available published data; the velocity of the moving medium in cms/sec. was adopted from the study of the Argentine data. The results are so far in harmony with the observed $\Delta H = s$ of Table 74, South Temperate zone, equivalent approximately to

s of the North Temperate zone, that we must admit that there is a close causal connection. The magnitude and direction of the deflecting magnetic vectors are so far in harmony with the convectional vectors, in all parts of the earth, that the subject will deserve to be further studied, especially in the determination of the wind vectors in the lower strata of the atmosphere.

The Aperiodic Magnetic Vectors Along the Meridians

According to Tables 66, 67, and Fig. 55, there are two principal regions of the absorption of the incoming solar radiation, the cirrus region and the cumulus region, in both of which there is transformation of energy into heat or into electric ions. The consequences of such ionization have been studied in the cumulus region, in the induced periodic diurnal or low-level variations of the magnetic field. It remains to give some account of the effects of the ionization in the cirrus region upon the earth's normal magnetic field. In order to analyze this subject the hourly variations are eliminated by taking the mean daily values of H the horizontal force, D the declination, and V the vertical force, as commonly published. As an example of the world-wide correlation of these daily movements of the magnetic field the horizontal force is transcribed in scale divisions, or units of force, for Greenwich, Toronto, Singapore, St. Helena, Cape of Good Hope, Hobarton, very widely separated in latitude and longitude. It is seen that substantially the same sort of variations, $+ \Delta H$, $- \Delta H$, occur nearly simultaneously all over the earth. Similarly, there are $+ \Delta D$, $- \Delta D$, $+ \Delta V$, $- \Delta V$ variations occurring from day to day. These rectangular variations must first be all transformed to C. G. S. units $d x$, $d y$, $d z$, and these are to be combined in polar co-ordinates s, α, β, which give the magnetic deflecting vectors disturbing the normal field of the earth. In computing ΔH, ΔD, ΔV from day to day, since there is an incessant secular or long-period variation of H, D, V, it is necessary to secure a proper base line, with appropriate slope, to which ΔH, ΔD, ΔV may be referred. This is best done by constructing the 10-day consecutive means, H_0, D_0, V_0 for every day of the year, so that the consecutive mean plus the variation is the observed value, $H_0 + \Delta H = H$, $D_0 + \Delta D = D$, $V_0 + \Delta V = V$. In passing the dates of excessive magnetic storms, it is proper to substitute a minimum $\Delta H_{min} = 0.00025$ C. G. S. It has been proposed to obtain the normal field by taking out the "quiet" days for

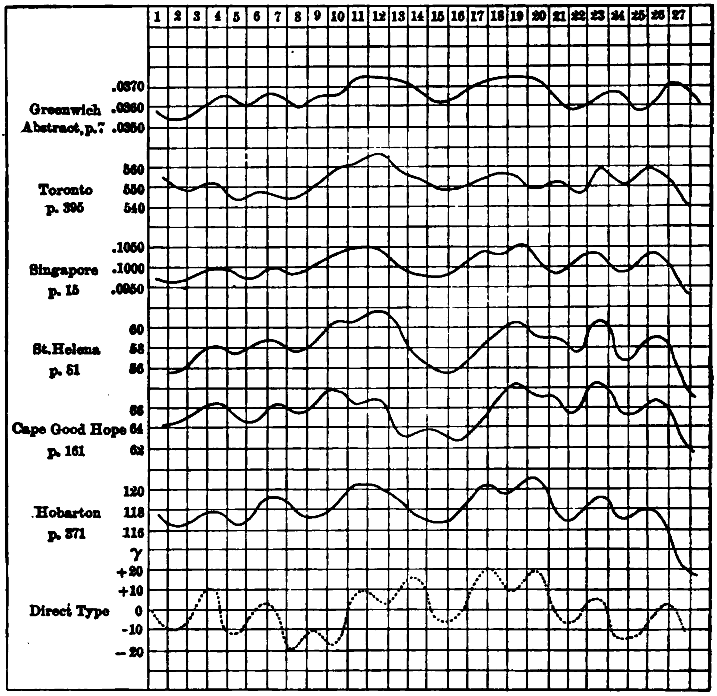


FIG. 65. Example of the variations of the H-component of the magnetic field in all latitudes, and showing the derived normal type curve in the 26.68-day period (direct), beginning August 3.58, 1845. $\gamma = 0.00001$ C. G. S. unit.

each month, and computing the means from these selected days. Unfortunately the fact that a day is "quiet" does not guarantee that the day is near the normal, because "quiet" days are as likely to run on one side of the normal as are the rough or moderately disturbed days. The best daily and monthly means are derived by taking all the observations as they occur, except that all variations greater than 0.00025 C. G. S. shall be counted at that value for the sake of taking out the consecutive means. There has been great confusion in the instrumental data, in the manner of discussing the variations, and in the interpretation of the results.

The computed magnetic vectors have been found to possess

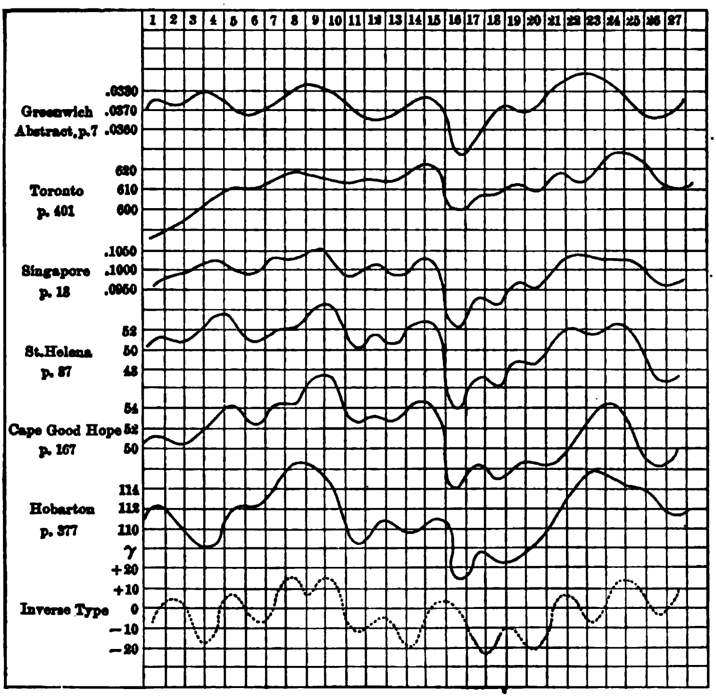


FIG. 66. Example of the variations of the H-component in the 26.68-day period (inverse), beginning November 18.30, 1845. A careful study of these curves, 1843-1905, shows that this type (reversed) occurs semiannually. Direct type, Feb. 1-April 20 and July 15–Oct. 15; inverse, April 20–July 15 and Oct. 15–Feb. 1.

interesting and important characteristics. Bulletin No. 21, U. S. W. B., 1898, contains a full explanation of these data, to which the reader is referred. (1) These vectors of deflection are generally closely confined to the magnetic meridians, and depend chiefly upon ΔH, ΔV. According to the latitude of the station the vector, s, α, β has well-defined lengths and vertical angles, similar to those seen in Figs. 68, 69. Corresponding with the waves in Figs. 65 66 these vectors point first south-

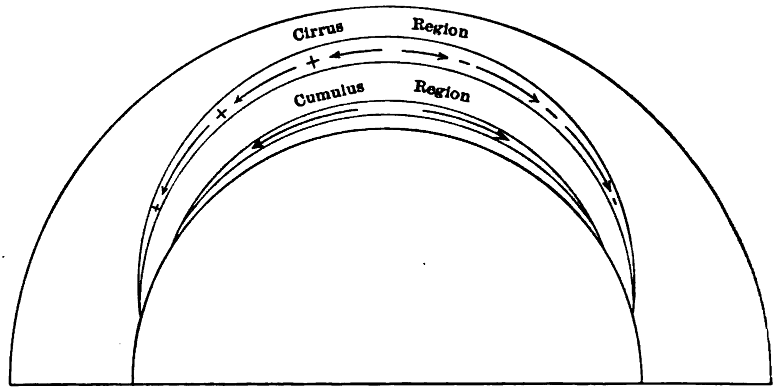


FIG. 67. Generation of the ions in the cirrus and cumulus regions. They flow alternately toward the north and south poles.

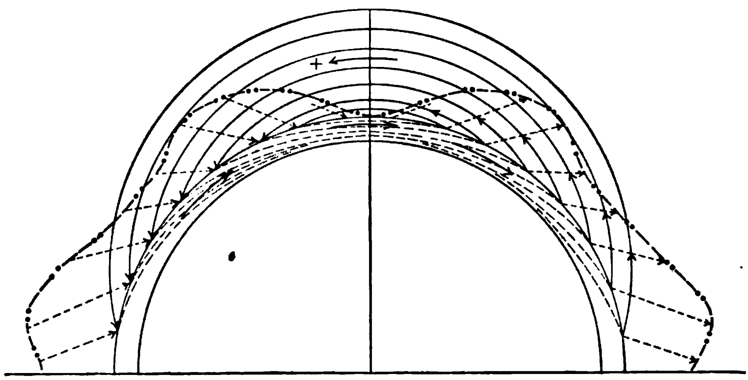


FIG. 68. Closed circuit for southward magnetic vectors. The + ions flow toward the north pole.

ward, Fig. 68, and then northward, Fig. 69, alternating about every three days. (2) A very extensive study of these vectors for the years 1843–1910 shows that they have a well-defined period of recurrence, on the average 26.68 days in length, and with two types, the direct as in Fig. 65, and the inverse, Fig. 66, the relative intensity from day to day being shown in the lower section of each figure. The recurrences are complicated with many irregularities, but the periodic action is unmistakable and corresponds with the synodic period of the rotation of the sun on its axis, as observed in the equatorial zone. The inference follows that these magnetic meridian deflecting vectors depend upon certain variations in the solar radiation, distributed in

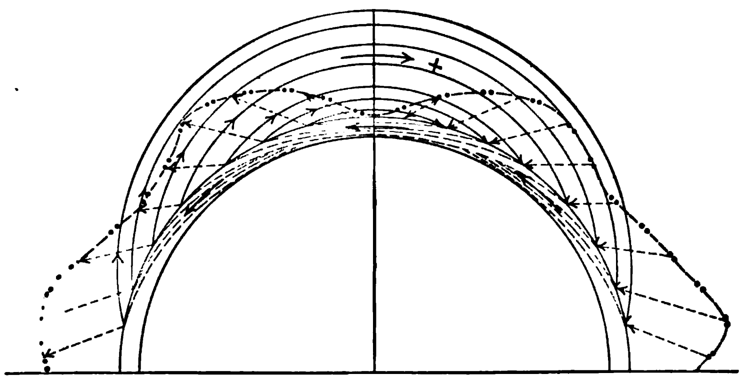


FIG. 69. Closed circuit for northward magnetic vectors. The + ions flow toward the south pole.

solar longitudes in such a manner that certain areas of the solar surface emit stronger radiations than do others in different longitudes. The equatorial period, 26.68 days, is exactly the same as the period determined from numerous direct observations on the sun spots, the faculæ, and certain spectrum lines. From a least square solution of the magnetic data, an ephemeris was constructed on the period 26.679 days, and epoch, June 13.72, 1887.

(3) The periodic reversal of the type curve occurs in semi-annual periods, as determined by the records, 1841–1894. Take the successive periods by years and match the type curve with the observations as in Figs. 65 and 66.

TABLE 76

SOLAR MAGNETIC EPHEMERIS, PERIOD 26.679 DAYS, EPOCH JUNE 13.72, 1887

1840	Jan. 16.87	1870	Jan. 24.05	1900	Jan. 5.55
41	Jan. 24.38	71	Jan. 5.88	01	Jan. 14.06
42	Jan. 6.21	72	Jan. 14.39	02	Jan. 22.57
43	Jan. 14.72	73	Jan. 21.90	03	Jan. 4.40
44	Jan. 23.23	74	Jan. 3.73	04	Jan. 12.91
45	Jan. 4.06	75	Jan. 12.24	05	Jan. 20.42
46	Jan. 12.57	76	Jan. 20.75	06	Jan. 2.25
47	Jan. 21.08	77	Jan. 1.59	07	Jan. 10.76
48	Jan. 2.91	78	Jan. 10.09	08	Jan. 19.27
49	Jan. 10.42	79	Jan. 18.60	09	Jan. 26.78
1850	Jan. 18.93	1880	Jan. 27.11	1910	Jan. 8.61
51	Jan. 27.44	81	Jan. 7.94	11	Jan. 17.12
52	Jan. 9.27	82	Jan. 16.45	12	Jan. 25.63
53	Jan. 16.78	83	Jan. 24.96	13	Jan. 6.46
54	Jan. 25.29	84	Jan. 6.79	14	Jan. 14.97
55	Jan. 7.12	85	Jan. 14.30	15	Jan. 23.48
56	Jan. 15.63	86	Jan. 22.81	16	Jan. 5.31
57	Jan. 23.14	Epoch 87	Jan. 4.64	17	Jan. 12.82
58	Jan. 4.97	88	Jan. 13.15	18	Jan. 21.33
59	Jan. 13.48	89	Jan. 20.66	19	Jan. 3.17
1860	Jan. 21.99	1890	Jan. 2.49	1920	Jan. 11.67
61	Jan. 2.82	91	Jan. 11.00	21	Jan. 19.18
62	Jan. 11.33	92	Jan. 19.51		
63	Jan. 19.84	93	Jan. 27.02		
64	Jan. 1.67	94	Jan. 8.85		
65	Jan. 9.18	95	Jan. 17.36		
66	Jan. 17.69	96	Jan. 25.87		
67	Jan. 26.20	97	Jan. 6.70		
68	Jan. 8.03	98	Jan. 15.21		
69	Jan. 15.54	99	Jan. 23.72		

THE SEMIANNUAL REVERSAL OF THE DIRECT AND INVERSE TYPES

Period	1	2	3	4	5	6	7	8	9	10	11	12	13	14
D	17	32	42	41	18	7	13	33	43	41	30	16	10	12
I	37	22	12	13	36	47	41	21	11	13	24	38	44	42
Type	*I*	*D*	$D_{max.}$	*D*	*I*	$I_{max.}$	*I*	*D*	$D_{max.}$	*D*	*D*	*I*	$I_{max.}$	*I*

The direct type prevails annually, February 1 to April 20.
The direct type prevails annually, July 15 to October 15.
The inverse type prevails annually, April 20 to July 15.

The inverse type prevails annually, October 15 to February 1.

These facts of periodic action from the sun in the equatorial period of 26.68 days, together with the semiannual inversion of the type, indicate that the problem of the solar radiation at the sun, and in its effects throughout the earth's atmosphere is an exceedingly complex phenomenon, which will require extensive researches of various kinds.

By way of suggestion it may be seen on Fig. 67 that if the incoming radiation transforms a part of its energy in the cirrus region into positive (+) and negative (−) ions, it may be supposed that they seek the poles of the earth in opposite directions, as (+) to the north pole and the (−) to the south pole, completing their circuit through the outer shell of the earth. This generates the magnetic vector system pointing southward, and the corresponding earth electric currents; at another time the (+) ions seek the south pole and the (−) ions the north pole, thus producing the northward vectors, and the corresponding earth electric currents. This reversal of direction from time to time depends upon the physical condition of the atmosphere as a conducting medium for the ions, its congestion of ions, its accumulation of ice and vapors, producing the magnetic vectors, auroras, magnetic storms, electric currents, in the well-known conditions as observed. The energy expended at the earth is that transformed from the solar radiation; it is inexhaustible in amount, and depends for the observed aperiodic irregularities upon the prevailing states of the solar and terrestrial atmospheres.

The Synchronous Annual Variations of the Solar and the Terrestrial Elements

The possibility of a scientific forecast of the type of weather likely to prevail in a large country as the United States or Argentina, whether the coming year is to be rainy and cool, or dry and warm, depends upon the establishment of the following two propositions: (1) The radiation output of the sun is a variable quantity, as 4 or 5 per cent. each side of the mean; (2) The meteorological elements, temperature, barometric and vapor pressures, and the precipitation synchronize with the solar changes in their annual variations. The evidence at present enables us to affirm that both are true, and that the synchronism exists, though in a very complex form, because the prevailing local conditions depend primarily upon the general circulation, and therefore only indirectly upon the solar variations. It is not possible in this place to do more than summarize the general principles that have been established in a research extending over twenty years, and embracing the available solar and terrestrial data. The first task is to procure *homogeneous* material of the several observed quantities, extending over a long series of years, sun-spot frequencies, solar-prominence frequencies, amplitudes of the terrestrial magnetic field, barometric pressures in all parts of the world, temperatures, and vapor pressures in all countries, precipitation in many districts, direct observations of the solar radiation in calories per square centimeter per minute. Unfortunately the difficulties of securing such homogeneous data of any of these elements is greatly complicated by the irregular and inconsistent methods that have been employed by meteorologists. In consequence of the necessity of substituting a few selected hours of observing for the twenty-four hours of each day, it is necessary to reduce the means from selected hours to the mean of twenty-four hours, which involves a long, special research for each country. The selected hours are different in different countries; the series are broken by changes in the selected hours in consequence of some administrative requirement; the corrections change from place to place when the same hour is made the basis of the work, as where the 75th meridian of the United States is made the hour of observing, which involves a range of three hours locally between the Atlantic and the Pacific States; or where the Greenwich noon is the basis of simultaneous world observations, involving variations up to twelve hours in local conditions; the altitudes and locations of the instruments in great cities have been not infrequently changed, and the instrumental equipment and the methods of computing have never been uniform for the long series. It is necessary to overcome these obstacles by setting aside a reasonable number of permanent stations for long series of fundamental work in meteorology, just as astronomers dedicate certain observatories to fundamental star places upon which the National Ephemerides are based. Cordoba, in

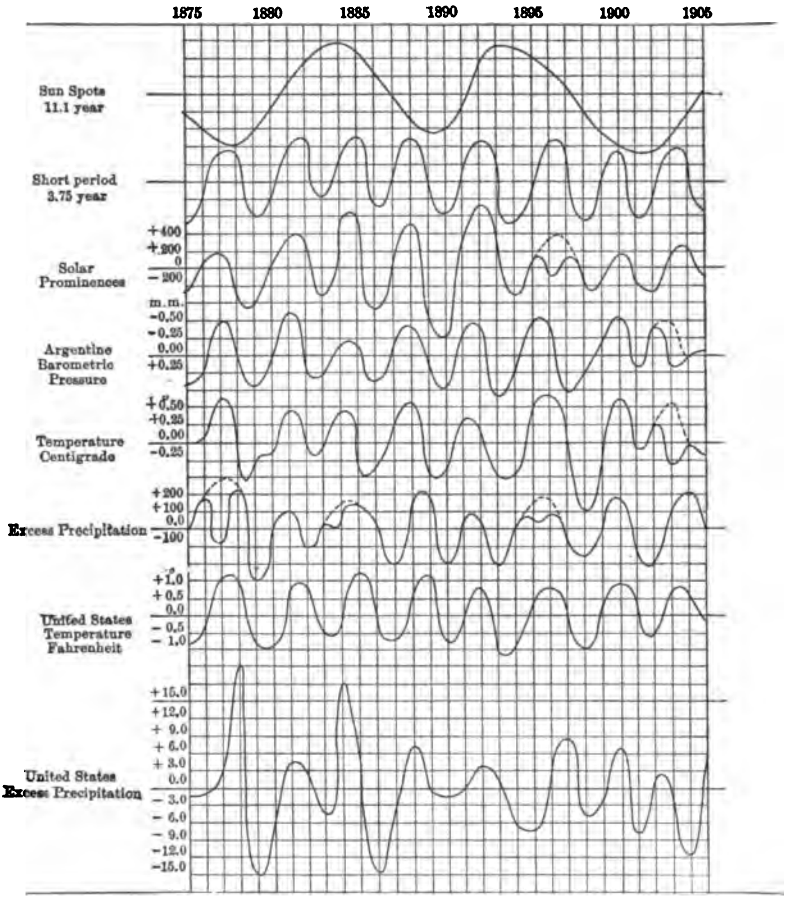


FIG. 70. Synchronism between the annual variations of the solar and terrestrial phenomena.

Argentina, is such a first-class meteorological station, because, since 1870, the instruments have had the same natural exposure, and practically the same apparatus has been used continuously, subjected to numerous tests for normality. There is no station

TABLE 77

THE SYNCHRONISM BETWEEN THE SOLAR AND THE TERRESTRIAL ANNUAL VARIATIONS OF THE METEOROLOGICAL ELEMENTS

Year	Sun-spots		Prominences		Horizont. Mag. Am.		Argentina				United States		
	C M	R	C M	R	C M	R	Δ B	Δ T	Δ e	Prec.	Δ B	Δ T	Prec.
							mm.	°C	mm.	mm.	inch	°F.	inch
1872	720	+500	1800	+845	2088	+679	−0.94	+0.21	+0.29	− 76	.....	...	− 9.60
73	645	+151	1700	+353	2054	−251	− .58	+ .15	+ .72	+ 32	.....	...	+ 9.35
74	578	− 43	1569	−154	1894	+168	+ .05	− .58	− .48	+ 78	.....	...	− 7.46
75	364	−159	1188	−382	1630	− 61	+ .47	+ .09	− .31	− 50	.....	...	− 0.89
76	213	− 77	865	+ 11	1681	−358	+ .04	+ .04	+ .26	+ 94	.....	...	− 0.99
77	120	+ 27	695	+ 47	1715	−320	− .52	+ .72	+ .70	−104	.....	...	+ 2.65
78	177	−136	791	−353	1948	+110	+ .25	− .62	+ .20	+252	−0.021	+0.6	+22.11
79	260	−188	1015	−454	2256	− 23	+ .33	− .14	− .51	−327	+ .005	+0.4	−15.18
1880	373	+ 14	1332	+ 6	2729	+ 1	− .26	− .22	− .05	− 69	+ .023	−1.4	− 6.15
81	518	+133	1064	+394	2946	− 80	− .41	+ .37	+ .46	+107	− .010	+0.4	+ 4.47
82	656	+ 60	2063	+264	2144	+613	+ .22	− .18	− .49	−108	+ .008	−0.2	+ 0.60
83	704	+ 61	2252	−456	3193	− 47	+ .20	− .01	+ .17	+ 73	+ .021	−0.3	− 5.39
84	634	+127	2170	+686	3204	+ 19	− .18	+ .37	+ .36	− 22	− .024	−0.9	+18.36
85	523	+103	2080	+204	2920	+ 53	+ .14	− .48	− .19	+174	− .011	+1.6	− 0.81
86	386	− 81	2098	−513	2779	+142	+ .33	− .25	− .36	− 87	+ .009	−0.3	−14.84
87	249	− 92	1671	+206	2569	−233	− .17	+ .35	− .11	−204	+ .010	−0.6	− 5.76
88	140	− 59	1352	+585	2298	+144	− .43	+ .54	+ .48	+ 83	− .014	+0.6	+ 7.59
89	165	− 90	1402	−678	2220	− 48	+ .11	− .45	+ .15	+217	− .009	+0.6	− 0.31
1890	309	−224	1521	−836	2473	−854	+ .50	− .26	− .77	−186	+ .016	−0.6	− 0.44
91	496	− 69	1592	+244	2613	− 82	− .57	+ .29	+ .49	+108	− .002	+0.3	− 0.51
92	669	+207	1740	+732	2795	+804	+ .04	− .10	+ .08	− 31	− .003	+0.4	+ 3.69
93	805	+214	1931	+310	2993	+151	+ .65	− .42	−1.03	−185	+ .003	−1.0	+ 0.33
94	820	+116	1806	−339	3073	+ 7	+ .05	− .43	− .29	+ 60	+ .006	−0.5	− 5.31
95	708	+ 60	1534	+106	2806	−195	− .47	+ .48	+ .15	+ 35	.000	0.0	− 7.48
96	568	− 67	1230	− 19	2650	+280	− .08	+ .91	+ .83	+ 21	− .007	+0.7	+ 2.15
97	410	− 95	1087	+ 76	2441	−177	+ .52	− .14	+ .09	− 17	+ .005	+0.2	+ 8.38
98	279	+ 42	817	− 96	2236	+130	+ .09	−1.10	− .91	−112	+ .001	−0.4	− 5.39
99	185	− 40	604	−106	1925	+107	− .45	+ .21	+ .28	+129	− .003	−0.5	− 2.80
1900	135	− 21	398	+145	1765	−176	− .25	+ .59	+ .61	+169	− .002	+1.0	+ 6.63
01	129	− 96	295	−150	1717	−343	+ .11	− .04	− .67	−211	+ .008	+0.3	− 7.98
02	201	−140	305	−222	1853	−389	− .44	+ .16	+ .11	−134	− .017	−0.7	+ 2.16
03	330	− 38	300	− 96	1896	+230	+ .22	− .32	+ .17	+ 76	+ .014	−0.5	− 0.85
04	453	+ 51	400	+150	2000	−213	+ .01	− .03	+ .06	+229	+ .008	+0.7	−11.53
05	550	+212	600	− 81	2180	+ 14	− .08	− .19	+ .12	+ 41	− .009	0.0	+ 7.81
06	650	− 4	....	...	....	...	− .17	+ .74	− .01	−197	.....	...	+19.11
07	.....	...	....	...	....	...	− .08	− .02	+ .15	− 5	.....	...	+ 0.26
08	.....	...	....	...	....	...	+ .41	− .26	− .17	+ 26	.....	...	.....
09	.....	...	....	...	....	...	+ .20	− .01	− .47	−115	.....	...	.....
1910	.....	...	....	...	....	...	− .16	+ .45	− .20	− 66	.....	...	.....
11	.....	...	....	...	....	...	− .11	− .46	− .07	+178	.....	...	.....

C M = consecutive mean; *R* = residual; *C M* + *R* = Observed.
Δ *B* = variation of barometric pressure; Δ *T* = variation of temperature.
Δ *e* = variation of the vapor pressure; Prec. = excess of precipitation.

in the United States that compares with it, because of changes of one kind or another in the instrumental conditions or the hours of observing. The writer spent many years in adjusting the imperfect observations in the United States, and finally produced a set of series of Pressure, Temperature, Vapor Pressure, and Precipitation, that are fairly homogeneous, and form the fundamental basis from which the annual variations may be computed. Similar reductions to homogeneous data are being made in Argentina, and in other countries, and in time it is hoped that world-wide comparable series of reduced observations may be made accessible to the scientific public.

Table 77 and Fig. 70 contain a series of examples of the results of such a comparison of the solar and terrestrial annual data, enough to give the reader a fair idea of the possibilities of this important subject. The sun-spots are from Wolfer's data, and the consecutive means C. M. added to the residuals R produce the observed annual means O, C. M. + R = O; the solar prominence frequencies are from the data of the Italian observers; the amplitudes of the horizontal magnetic force were compiled from several European observatories: the Argentine Meteorológica can be found in Bulletin No. 1, Oficina Meteorologica Argentina, 1911; the United States data may be found in the Barometry Report, U. S. W. B., 1902; Temperatures and Vapor Pressures in Bulletin S, 1909; Temperature and Precipitation Normals, Bulletin R, 1908; Temperature Departures, Bulletin U, 1911; Climatological Summary in 106 sections, Bulletin W, 1912, all prepared under the writer's supervision. The consecutive means represent a long periodic cycle averaging 11.1 years in duration, but very irregular in length, as from 8 years to 14 years between certain maxima; the residuals represent a short periodic cycle averaging 3.75 years, but ranging between 3 years and 5 years. The first curves of Fig. 70 represent the 11.1 years and the 3.75 years cycles between 1875 and 1905. Following them are several curves for Argentina and the United States in the short period cycle. The synchronism in the short period is pronounced, in spite of certain irregularities, demonstrating the general fact that the sun has a variable output of radiation which persistently modifies the earth's circulation and climatic conditions. The barometric pressures and temperatures were studied in all parts of the world and the result summarized in *Monthly Weather Review*, October and November, 1903.

Barometric Pressure. The net work of barometric pressures for the world, taken for the annual variations, shows that the stations must be divided into two classes: (1) Those where the synchronism is direct between the pressure and the prominences, as surrounding the Indian Ocean, and those where the synchronism is inverse as in North and South America. Under the external impulse from the sun an increase of the annual radiation accelerates the general circulation in such a way that the pressure is simultaneously higher in certain large regions and lower in others. This is due to the fact that the total pressure of the earth's atmosphere is an invariable constant, so that if the pressure in one region is relatively high, that in another region is relatively low at the same time. The wandering cyclones and anticyclones, added to the more permanent centers of high and low pressures, should sum up to the same constant for the world. The oscillation of regional pressures is, therefore, a fundamental fact leading to an extensive study of the pressure conditions in various localities.

The Temperatures. Similar studies of the annual temperatures divide the stations into two groups, (1) Those in the Tropics with direct synchronism, (2) those in the Temperate zones, on the poleward side of the high-pressure belt, with inverse synchronism. There are many places of mixture or disintegrated effects which it is still difficult to classify. An increase of solar radiation increases the vertical convection of the Tropics, with increase of the surface temperature; this is followed by an increase of downflow in the Temperate zones, with an extension of the high areas and cooler temperatures. The temperature integral of the entire earth's atmosphere must be nearly a constant, or else the earth's rotation period of twenty-four hours would indicate variations of an astronomical value, which have never been detected in the observations.

Precipitation. The changes in the general and the local circulations, depending upon the solar variations, carry with them the rain-bearing currents, as from the oceans to the continents, and thence the annual amounts of the precipitation in the regions concerned. There are great irregularities in these precipitations from one region to another, from one year to another, and for the same station. The results from Argentina and the United States indicate clearly that the precipitation synchronizes with the solar variations, and that the variations are of large amounts, ranging through 400–500 millimeters in Argentina at the same station, and several inches in the United States.

Partial Formations. Fig. 70 shows that the annual crests in certain elements occasionally fail to form completely in the 3.75-year period, and for this cause irregularities appear in the series of curves. It is easy to see how this may occur in many cases by a sort of self-contradiction in natural causes and effects. Thus, if in a certain region the excess of solar radiation of the Tropics has produced higher temperatures, this has resulted in spreading a rain and cloud sheet over another region at a distance from it, both due to the same cause. This very cloud sheet, however, acts as a screen upon the *surface* temperatures, so that lower local temperatures are registered at the surface, while they are really higher above the cloud sheet. The rain currents may precipitate so much aqueous vapor on one side of a mountain range that the overflow on the other side is dryer than usual, so as to give opposite effects for the same efficient increase of circulation, excess in one region, and defect in another region. The observations of the solar prominences depend upon the number of clear days per month. Hence, an increase in solar radiation, following an increase in the frequency of the prominences, may locally produce a cloud sheet, and hence a lower annual count in the number of the prominences. It is quite irrelevant to attempt to discredit the facts of synchronism, by presenting irregularities or inconsistencies in certain localities, unless the trouble is taken to understand the full series of causes and effects between solar action and final local conditions. Since opposite results, inversions of effects, are inevitable in terrestrial meteorology, from the same solar cause, it will be necessary to study carefully the history of each region, before attempting to arrive at any conclusions.

The magnetic field presents similar synchronous variations, as may be seen by plotting the amplitude curves. This element is very sensitive to many radiation and ionization influences, and it is our purpose to pursue the research into the function connecting these several elements.

The radiation in calories per square centimeter per second does not yet present annual variations which seem to be reliable. The cause of this result is seen in the section on radiation, and may be verified by studying the divergent annual values on Table 62.

The possibility of annual forecasts of the weather conditions is being tested in Argentina by projecting forward the normal 3.75-year curve from 1911 to 1915. The results for 1911, 1912, and 1913 are entirely successful, the precipitation being quite the same as indicated in Bulletin No. 1, O. M. A. It is certainly possible to make similar forecasts for the United States as to precipitation in different districts, wherever the sequence of the rainfall in each district is studied in relation to the fundamental solar 3.75-year period. Compare Abstract No. 3, U. S. W. B., 1909. with the data of Fig. 70.

The Aqueous Vapor in the Atmosphere

It is evident from the discussions on radiation, on cloud formation and precipitation, and on evaporation of aqueous vapor from areas of water, as in lakes and oceans, that the presence of aqueous vapor in the atmosphere is of primary significance. We can compute the number of grams of aqueous vapor per cubic meter of air, or per kilogram of air, according to convenience.

Grams of Aqueous Vapor in 1 *Cubic Meter of Saturated Air*

$$(742) \quad \mu = \rho_0 \frac{273}{T} \cdot \frac{287}{R} \cdot \frac{B}{760} \left(0.622 \frac{e}{B} + 0.235 \frac{e^2}{B^2}\right)$$

The full form most used at all elevations above the sea level.

By substituting the observed vapor pressure e, the temperature T, the barometric pressure B, at any other point, the corresponding μ can be computed. Extensive tables have been prepared for μ where T ranges from $-50°$ C. to $+50°$ C., and B from 800 mm. to 20 mm.

Grams of Aqueous Vapor in One Kilogram of Saturated Air.

(743) $$\mu = 0.622 \frac{e_0}{B} + 0.235 \frac{e_0^2}{B^2}$$

e_0 is the saturated vapor pressure for temperatures ranging from $-50°$ C. to $+50°$ C.

When the air is not saturated the following formula serves: e_0 = the saturated vapor pressure, t = the dry-bulb temperature, t_1 = the wet-bulb temperature.

Vapor Pressure in Millimeters when the Air is not Saturated

(744) $$e = e_0 - 0.00066\, B\, (t - t_1) \left(1 + \frac{t_1}{873}\right).$$

Tables applicable to practical work may be found in Bulletin No. 2, Oficina Meteorologica Argentina, 1912.

In the free air the aqueous vapor is distributed approximately by Hann's formula,

(745) $$e = e_0\, 10^{-\frac{h}{6517}},$$

where h is the height in meters.

The Laws of the Evaporation of Water from Lakes, Pans, and Soils with Plants

The subject of the evaporation of water has been very extensively studied, and there is a large literature on the results. These, however, are unsatisfactory as concerns the terms and the coefficients of the proposed formulas. Another research was undertaken by the writer in 1907 for the U. S. Weather Bureau, at Reno, Nevada, where the proper type of formula was determined; it was continued in 1908 at Indio and Mecca, So. California, and at the Salton Sea, 1909, 1910, in co-operation with numerous stations in various parts of the United States, during which the coefficients were approximately computed; the work was continued in 1911, 1912, at Cordoba, Argentina, and extended to include evaporation from soils, and soils with plants of different kinds, and the final coefficients with the necessary working tables for the computations were constructed. The results of this work are summarized in Bulletin No. 2, Argentine Meteorological Office, 1912. Several special pieces of apparatus have been invented: Bigelow's micrometer hook gage for measuring the water height, Bigelow's dial gage for measuring the water height in soil tanks, Wilcken's self-registering apparatus for continuous records of every position of the water surface. The principal difficulty in arriving at conclusions has been due to the necessity of using pans for evaporation, in which case the wind in blowing over the pan greatly complicates the action of the evaporation. Pans of different sizes in the same wind evaporate different amounts during the same interval of time, because the wind carries away the evaporated vapor at different rates, according to the size of the pan, and thus produces a varying mixture of dry air and vapor. A large body of water in a wind, and a small pan in a calm, produce the same effect as an evaporating medium, because the vapor is actually the same in density near the water on a lake in a wind, which merely transports it from place to place without really removing it, as in a calm air over a small pan. The result is that lakes evaporate only at about two-thirds the rate from pans near by in moderate winds. In certain places it was found that a small pan evaporates three times as much water as does a lake in the neighborhood. For example, there were three towers built in the Salton Sea, No. 2 near the shore, No. 3 about half-mile from No. 2, and No. 4

TABLE 78

EXAMPLES OF THE ANNUAL EVAPORATION AT THE SALTON SEA

Tower No. 1.	Pan (5), 40 feet above the desert,	195 inches.
Tower No. 1.	Pan (1), on the ground of the desert,	165 "
Tower No. 2.	Pan (5), 40 feet above the water,	138 "
Tower No. 2.	Pan (1), 2 feet above the water,	109 "
Tower No. 4.	Pan (5), 40 feet above the water,	140 "
Tower No. 4.	Pan (1), 2 feet above the water,	106 "
The evaporation from the Salton Sea itself,		72 "

about one mile from the shore, while No. 1 was 1,500 feet inland from the sea in the desert. These towers carried pans near the surface of the water and at every 10 feet up to 40 feet above the water. The evaporation for a year was as follows at several pans, as summarized in Table 78.

The evaporation was registered at other stations from pans of different sizes on the ground, and on a stand 10 feet high, of which annual examples follow.

EVAPORATION AS RECORDED IN SEVERAL PLACES

Station	Indio		Mecca		Brawley		Mammoth	
Height	Ground	10 feet	Ground	10 feet	Ground	10 feet	Ground	10 feet
Size of pan	6 feet	2 feet	6 feet	2 feet	6 feet	2 feet	6 feet	2 feet
Evaporation	119	200	108	170	104	164	126	179

Station	N. Yakima		Cincinnati		Birmingham		Lake Tahoe	Lake Kechess
Height	Ground	10 feet	Water	10 feet	Water	10 feet	2 feet	10 feet
Size of pan	4 feet	3 feet	4 feet	3 feet	4 feet	2 feet	4 feet	3 feet
Evaporation	68	86	46	62	51	64	42	33

The formulas that have been found to be adequate to follow the course of evaporation in all climates, that is, in all conditions of temperature, vapor pressure, and wind velocity, are as follows:

Hours of observation for 4-hour intervals (2, 6, 10) A.M. (2, 6, 10) P.M.

t = the temperature of the dry bulb on the whirling psychrometer as usually employed.

t_1 = the temperature of the wet-bulb thermometer.

e_d = the computed vapor pressure at the dew point d.

S = the temperature of the water surface.

e_s = the computed vapor pressure at saturation S.

$\frac{de}{dS}$ = the rate of change of the vapor pressure with the temperature change of the water.

w = the velocity of the wind in kilometers per hour, derived from the successive anemometer readings.

Formula of Evaporation from Large Water Surfaces

$$(746) \quad \frac{E_0}{\text{4-hours}} = 0.0230 \frac{e_s}{e_d} \frac{de}{dS} (1 + 0.084\, w); \text{ (Argentine anemometer).}$$

Formula of Evaporation from Pans of Different Areas

$$(747) \quad \frac{E_0}{\text{4-hours}} = 0.0230\, F(w) \frac{e_s}{e_d} \frac{de}{dS} (1 + 0.084\, w).$$

$F(w)$ = a factor depending on the area of the pan, which varies with the wind velocity up to about 10 kilometers per hour. $F_1(w)$ applies to the Dines' system of wind velocities, used by the Argentine Meteorological Office, and a pan of 1.0 meter^2 area; $F_2(w)$ to the same wind system and a 0.5 m^2 area pan; $F_3(w)$ to the wind system used in the United States, wherein the same wind velocity is recorded higher in the ratio 1.21 to 1.00, and a 1.0 m^2 pan; $F_4(w)$ to the U. S. wind system and a pan 1.17 m^2 area or 4 feet in diameter; $F_5(w)$ to the U. S. wind system and a pan 0.29 m^2 area, or 2 feet in diameter. There are two wind systems in use: (1) that based upon the Dines' pressure-velocity, and (2) that based upon the whirling machine velocities. Thus the anemometers by Casella, Negretti, and Zambra, U. S. Weather Bureau Freiz, Richard, are approximately in agreement together, but they are about 20 per cent higher than the Dines, Hess of the Oficina Meteorológica Argentina, Munro, and Tschau system of anemometers. Marvin's table of corrections to the Robinson anemometer gives about 20 per cent correction to reduce from the indicated to the true wind velocity, *Monthly Weather Review*, October, 1906, Table 64, so that the first group becomes equivalent to the second group after making this reduction. Unfortunately, it is customary to omit these reductions, so that the published wind velocities of the United States, and other countries using the above-mentioned anemometers, are about 20 per cent too great. It is indispensable in evaporation reductions that the coefficients should be adjusted to correct wind velocities. For this purpose the following factors $F(w)$ are introduced into the working Tables:

TABLE 79

THE FACTORS $F(w)$ FOR ADJUSTING THE EFFECTS OF THE WIND VELOCITIES FOR PANS OF DIFFERENT AREAS

System	Argentina		United States		
Wind Velocity	1.0 m² $F_1(w)$	0.5 m² $F_2(w)$	1.0 m² $F_3(w)$	1.17 m² $F_4(w)$	0.29 m² $F_5(w)$
$w =$ 0	1.000	1.000	1.000	1.000	1.000
1	1.150	1.148	1.120	1.150	1.160
2	1.265	1.274	1.240	1.212	1.290
3	1.376	1.392	1.320	1.289	1.410
4	1.463	1.493	1.400	1.367	1.520
5	1.542	1.592	1.480	1.433	1.630
6	1.600	1.667	1.540	1.480	1.710
7	1.617	1.712	1.590	1.523	1.760
8	1.627	1.746	1.615	1.542	1.810
9	1.629	1.762	1.623	1.552	1.830
10	1.629	1.777	1.629	1.561	1.840
15	1.629	1.782	1.629	1.561	1.850
20	1.629	1.782	1.629	1.561	1.850
25	1.629	1.782	1.629	1.561	1.850
30	1.629	1.782	1.629	1.561	1.850

The complete tables for evaporation computations may be found in Bulletin No. 2, Argentine Meteorological Office, 1912.

It has been found that about 90 per cent of the computed results are less than 0.30 mm. from those as observed. This difference includes the errors of measurement as well as of computation. The computed difference from the observed amounts for entire months in Cordoba is about 4 millimeters, and the total difference for the year on one pan was − 3 millimeters, and on another pan − 10 millimeters, the total in the first case being 1,091 millimeters, and in the second case 1,945 millimeters. Our experience leads us to conclude that pans need not be employed in work on evaporation, but that computations are quite as accurate provided observations of the water temperature S, the vapor pressure of the air e_d, and the velocity of the wind w are made. As it is impossible to float pans on large bodies of water, lakes and reservoirs, except under restricted conditions that injure the observations, it becomes necessary to dispense with pans entirely and depend upon simple computations. There are several methods of abbreviation for computing the mean monthly amounts of the evaporation from lakes and reservoirs, which make the computations an insignificant labor.

Studies on evaporation from soil, sand, soil planted with alfalfa, wheat, barley, beans, have been carried on, which analyze successfully the amount of water lost under all conditions throughout the year, from soils by themselves, and from the plants by themselves. Thus, the transpiration of plants is subject to accurate measurement and analysis, and the results, when sufficiently verified, will be of great value to meteorological agriculturalists and botanists.

The Polarization of Sunlight in the Atmosphere

Common sunlight vibrates indifferently in every plane perpendicular to its wave front, but when it falls upon any object, large relatively to its wave length, a portion of the light is refracted and a portion reflected in the plane of incidence containing the incident and reflected rays. The vibrations in the reflected ray become at least partially constrained to vibrate parallel to the surface of reflection, and it is plane polarized. The plane of the polarization is at right angles to the plane of vibration, and therefore contains the incident and the reflected rays. If an observer looks at any point in the sky he will receive certain reflected rays that have proceeded from the sun to the reflecting particle and to the eye, this plane being the plane of polarization, and the vibrations are at right angles to it. If the polarization is partial, and the motion circular, elliptical, or of any other figure, components of plane polarized light may be constructed for this plane and another at right angles to it, so that partially polarized plane vibrations in two directions at right angles may more or less neutralize each other between the limits 0 per cent and 100 per cent. A turbid medium, such as air mixed with small solid particles of dust, ice, or even molecules, whose diameters are small relative to the wave lengths of light, scatters and polarizes light by Rayleigh's Laws, in which

β = the angle of departure from the line of incidence for the reflected ray.

(748) Intensity of scattering $= 1 + \cos^2 \beta$.

(749) Fraction of light polarized $= \dfrac{\sin^2 \beta}{1 + \cos^2 \beta}$.

Hence the maximum scattering at $\beta = 90°$ from the sun is twice as much as in the direction of the sun, $\beta = 0$; the amount of polarization is 100 per cent at $\beta = 90°$ from the sun, and it is 0 per cent at $\beta = 0°$ in the direction of the sun. In the atmosphere with the sun on the horizon, as at the equinox in the east, the maximum polarization is in the zenith, and in the vertical plane passing through the zenith and the north and south points. If the solar point is east, the antisolar point is west; as the sun rises the antisolar point sinks below the horizon; as the sun moves to any other usual point the plane of polarization is that which includes the sun, the point of reflection in the sky, and the eye of the observer, the vibrations being generally at right angles to this plane.

Besides the primary scattering and polarization on the small particles in a turbid atmosphere, it is found that the light is only partially polarized, so that a secondary polarization exists at right angles to the primary, primary and secondary vibrations, and polarizations at right angles to each other, thus tending more or less to complete neutralization of plane polarized light as the primary and secondary components approach equality. There are several such points of neutralization: Babinet's neutral point about 15° to 25° above the solar point, Brewster's neutral point about the same distance below as the sun rises above the horizon, and Arago's neutral point about 15° to 25° above the antisolar point when the sun is on the horizon. The positions of these points vary with the position of the sun in the heavens, and the relative turbidity of the atmosphere. Since the dust particles accumulate chiefly in the lower atmosphere, in a stratum less than two miles thick, there is an apparent ring of special turbidity close to the horizon, which causes the light to be horizontally polarized within a few degrees of the horizon. Generally, polarization is a maximum in the zenith, and diminishes to the north and south horizon points, and from these to the east point, for a sun in the east and on the horizon. There are numerous variations of these principal results, due to change in the intensity of solar light from radiation, and change in the contents of turbidity in the atmosphere.

The subject of polarization is discussed fully in "Tatsachen und Theorien der atmosphärischen Polarisation," Friedr. Busch and Chr. Jensen, 1911. The literature of the observations and discussions is very extensive during one hundred years.

The observations are made by a polarimeter, consisting of a grating of parallel bars and spaces, from which the light falls upon a Rochon prism which separates it into the ordinary and extraordinary rays. These fall upon a Nichol, and by its rotation, there is extinction, or flattening of the appearance of the field, at four angles of observation. Thus for four angles of observed extinction, the computation is of the following form.

Cordoba.	(1)	(2)	(3)	(4)	(2–1)	(4–3)	Mean	P%
Feb. 8, 1912,	11°.3	94°.0	190°.0	278°.0	82°.7	88°.0	85°.4	73.4%

A convenient table is prepared for obtaining the mean percentage of polarization $P\%$ from the mean angles (2–1) and (4–3), where (1), (2), (3), (4) are the successive readings.

Various relations have been traced out, such as:

(1) Movement of the altitude of the neutral points with the frequency of the number of the sun-spots.

(2) Minimum polarization at the time of maximum temperature and maximum convection.

(3) Maximum polarization in winter rather than summer.

(4) Water drops have little effect on the polarization at 90° from the sun; ice crystals and large particles in that direction decrease the polarization and increase the natural scattered light; light that is reflected from the earth's surface, or from snow areas, diminishes the polarized and increases the reflected common light.

The relations between these several terms have numerous interesting optical considerations, and they serve to measure to some extent the state of turbidity in the atmosphere, and hence have value in connection with the absorption and radiation of solar energy in the atmosphere.

The polarization at Daroca, Spain, August 19, 26, 1905, was relatively high following rains, but it often fell to 40 per cent or

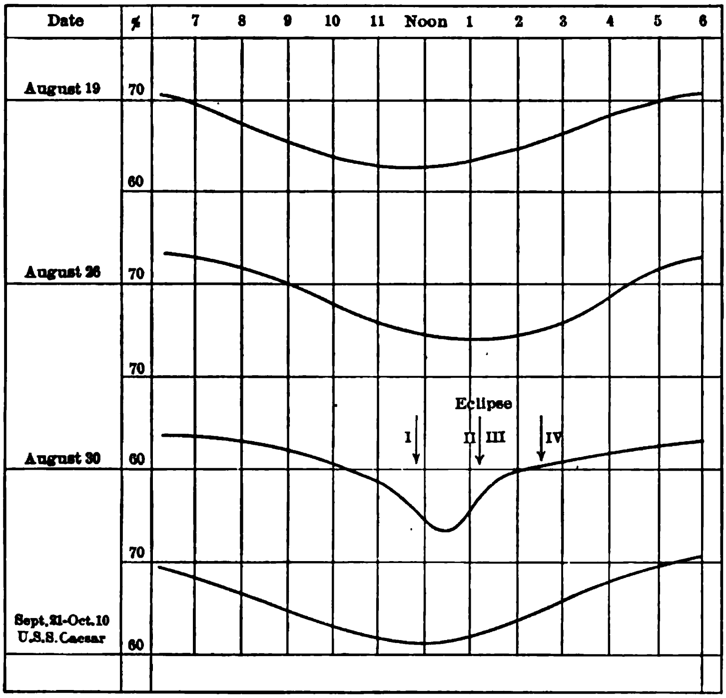


FIG. 71. Percentage of polarization of the sky light at Daroca, Spain, during August, and on the U. S. S. *Cæsar*, Sept. 21–Oct. 10, 1905.

50 per cent on account of fine dust in the air; that on August 30 shows the effect of the passage of the shadow of the solar eclipse, the sky having been thoroughly cleaned of dust by a rain on August 29; the observations on the U. S. S. *Cæsar* during the voyage from Gibraltar to Norfolk, frequently in the clear spaces between cumulus clouds, showed a normal high percentage of polarized light at 90° from the sun. Impurities from solid particles produce natural light by reflection, fine particles and gas molecules produce polarized light.

Solar Physics

It will be possible merely to summarize a few important points in the subject of solar physics, in this connection, because it is very extensive in amount, and in consequence of the fact that much of the theory is in a conjectural stage of development and is still indecisive.

(1) It is evident that the thermodynamic equations employed in the discussion of the earth's atmosphere are applicable to the sun's atmosphere, by changing the data in a proper manner. Thus, gravity becomes $G = g_0 \times 28.028$; the pressure on the photosphere is about five atmospheres, so that $P_0 = 5 \times G \rho_m B_n$; the temperature at the photosphere is apparently 7500° C.; at the top of the chromosphere or lower layers of the inner corona, 10″ arc = 7260000 meters, 6900°: and top of the inner corona, 35″ arc = 25410000 meters, 6500°. From these data for a hydrogen atmosphere, or a calcium atmosphere, the various thermodynamic terms can be computed as far as the united terms of the velocities' and radiations' energies.

(2) The probable velocities in the sun-spots, assuming that they are the stream lines of the funnel-shaped or the dumbbell-shaped vortices on the upper plane of reference, on a level with the layer of the photosphere, can be computed from the general dimensions of the penumbra and umbra, and checked to some extent by the spectroscopic observations on velocities. The vertical and the horizontal velocities in different layers of the sun's atmosphere are being studied with the prospect of ultimate success in a few years. It may be hoped that the radiation output from the sun, computed from such data, may be found to conform to the radiation energy at the earth as derived from the pyrheliometer and the bolometer records, but much research will be required to accomplish this result.

(3) The rotational velocity of the sun's atmosphere in different latitudes on the level of the photosphere, and in other higher layers as already determined, indicate a very complex kind of circulation, of an entirely different type from that in the earth's

atmosphere. The latter consists of a thin shell heated on the tropics, and acquiring an approximately steady type of equilibrium, as heretofore explained, while the sun has maximum velocity at the equator diminishing to the poles on the level of the photosphere, and increasing upward in all latitudes. Since the integral in every small column along a radius extended must conform to the gravity integral, which is the sum of the pressure, circulation across it, and radiation through it, there is an opportunity to determine these terms through an approximation by trials.

(4) Table 80 contains a convenient series of transformations between sidereal and synodic periods. Table 81 contains a collection of the observed synodic periods of rotation in different latitudes. Bigelow's data from the prominences refer to the higher levels of the sun's atmosphere, because they are seen projected above the chromosphere. The acceleration in the polar region over the velocities devised from spectrum displacement lines is probably correct, because the spectrum lines are all located at lower levels. Some of the Mt. Wilson data are in conformity with this result.

(5) The magnetic data at the earth, as already indicated, produce a synodic period of 26.68 days at the sun's equator, conforming closely to the general mean value 872′ or 26.58 days from the eight researches quoted. The Zeeman effect has been detected by Professor Hale in the sun-spots, due to the rotation of electric ions in the tube of the vortex. This proves that electric ions in circulation produce magnetic field at solar temperatures. Hence, the interior of the sun, if polarized into rotation filaments by its circulation, by rotation on its axis, and processes of radiation, is probably magnetized throughout its mass, in much the same way that the earth carries an internal and external magnetic field, though its interior is at a high temperature. There is evidence of such spherical magnetism in the shapes of the polar rays seen in the minimum activity of the coronal formation, where the observed rays from the sun, seen in projection, conform to the lines of force surrounding a sphere, supposing that they are generated chiefly in a polar ring about

TABLE 80

THE SIDEREAL AND SYNODIC PERIODS OF THE ROTATION OF THE SUN FOR CERTAIN ASSUMED DAILY ANGULAR VELOCITIES. $\frac{1}{T} - \frac{1}{E} = \frac{1}{S} = k - n$.

Daily Angular Velocity X	Daily Angular Velocity in Degrees ξ	Sidereal Period in Days T	Angular Velocity of the Sun in Days $\frac{1}{T} = k$	Angular Velocity of the Earth in Days $\frac{1}{E} = n$	Synodic Velocity of the Sun in Days $\frac{1}{S} = k - n$	Synodic Period in Days S
′	°					
700	11.67	30.86	.03241	.00274	.02967	33.70
705	11.75	30.64	.03264	"	.02990	33.44
710	11.83	30.42	.03287	"	.03013	33.19
715	11.92	30.21	.03310	"	.03036	32.94
720	12.00	30.00	.03333	"	.03059	32.69
725	12.08	29.79	.03357	"	.03083	32.44
730	12.17	29.59	.03380	"	.03106	32.20
735	12.25	29.39	.03403	"	.03129	31.96
740	12.33	29.19	.03426	"	.03152	31.73
745	12.42	28.99	.03449	"	.03175	31.50
750	12.50	28.80	.03472	"	.03198	31.28
755	12.58	28.61	.03495	"	.03221	31.05
760	12.67	28.42	.03518	"	.03244	30.83
765	12.75	28.24	.03542	"	.03268	30.61
770	12.83	28.05	.03565	"	.03291	30.39
775	12.92	27.87	.03588	"	.03314	30.18
780	13.00	27.69	.03611	"	.03337	29.97
785	13.08	27.52	.03634	"	.03360	29.76
790	13.17	27.34	.03657	"	.03383	29.56
795	13.25	27.17	.03681	"	.03307	29.35
800	13.33	27.00	.03704	"	.03430	29.15
805	13.42	26.83	.03727	"	.03453	28.96
810	13.50	26.67	.03750	"	.03476	28.77
815	13.58	26.50	.03773	"	.03499	28.58
820	13.67	26.34	.03796	"	.03522	28.39
825	13.75	26.18	.03819	"	.03545	28.20
830	13.83	26.02	.03843	"	.03569	28.01
835	13.92	25.87	.03867	"	.03592	27.83
840	14.00	25.71	.03889	"	.03615	27.66
845	14.08	25.56	.03912	"	.03638	27.49
850	14.17	25.41	.03935	"	.03661	27.32*
855	14.25	25.26	.03958	"	.03684	27.14
860	14.33	25.12	.03982	"	.03708	26.97
865	14.42	24.97	.04005	"	.03731	26.80
870	14.50	24.83	.04028	"	.03754	26.64†
875	14.58	24.69	.04051	"	.03777	26.48
880	14.67	24.55	.04074	"	.03800	26.32
885	14.75	24.41	.04097	"	.03823	26.16
890	14.83	24.27	.04120	"	.03846	26.00
895	14.92	24.13	.04144	"	.03870	25.84‡
900	15.00	24.00	.04167	"	.03893	25.69
930	15.50	23.22	.04306	"	.04032	24.80§

* Lat. .13°, spots. † Equator of sun. ‡ Terrestrial. § Hydrogen.

TABLE 81

OBSERVED SYNODIC DAILY ANGULAR VELOCITIES, X, FROM THE SUN-SPOTS, PROMINENCES, AND SPECTRUM LINES

Latitude ϕ	Carrington Spots	Spörer Spots	Faye Spots	Tisserand Spots	Bigelow High Level Prominences	Dunér Spectrum	Halm Spectrum	Adams Spectrum	Means
90°	...	...	...	...	788′	616′	732′	705′	710′
85	...	...	...	...	790	625	733	706	714
80	...	...	...	...	793	634	734	706	717
75	...	...	...	...	795	642	736	709	721
70	...	...	...	...	799	656	741	714	728
65	...	...	...	...	804	674	745	721	736
60	...	...	...	...	809	690	756	732	747
55	...	...	...	...	815	714	768	740	759
50	...	...	...	...	824	740	780	752	774
45	...	...	...	...	832	768	792	768	790
40	789′	810′	785′	793′	837	796	800	780	799
35	803	814	801	806	840	819	810	798	811
30	816	819	815	819	842	833	822	813	822
25	828	825	829	830	845	848	838	830	834
20	840	833	840	840	846	860	850	844	844
15	849	842	850	847	852	870	858	855	853
10	857	853	856	853	859	880	866	866	861
5	863	864	861	857	866	885	872	874	868
0	865	877	862	858	869	888	876	882	872

23 degrees distant from the coronal pole. This system has been traced from one epoch to another, through several eclipses from 1878 to 1905 at least, as if the synodic period 26.68 days was also fundamental in producing the aspect they present upon the series of eclipse photographs. A model was constructed of such a magnetic field, and turned by its astronomical co-ordinates into the required positions on the days of the several eclipses. The coincidence in position between the pole of the sun, pole of the earth, pole of the corona, and its stream lines as parts of a spherical magnetic field, are too striking to be overlooked. Superposed upon this deep-seated magnetic field, embracing the entire interior of the sun, is a strong electrostatic surface field with its rays in normal directions, and in many distorted positions. These two fields interplay among the forces of circulation and radiation to produce the numerous fantastic forms seen on the edge of the sun. Astron. Soc. Pac. No. 27, 1891.

The Spherical Astronomy of the Sun

It is necessary to give a brief account of the variable relations due to the rotation of the sun on its axis, and the revolution of the earth in its orbit about the sun. If the spherical conditions of Fig. 72 be transferred to a small rubber ball, it will greatly facilitate the study of this complicated branch of solar physics. The photographs of the sun give pictures which must be interpreted in terms of spherical co-ordinates, and this is a great labor of computation, where any large number of points are to be considered. Some mechanical devices have been used for securing heliocentric co-ordinates approximately, but for definitive work the micrometer measurements must be employed with accuracy. As the earth passes around the sun the aspect of the disk undergoes an annual periodic change which must be followed, and as the sun rotates on its axis the positions of the spots, faculæ, and prominences change from day to day. The following definitions and formulæ can be very readily verified from the diagram, and by Chauvenet's treatise on Spherical Trigonometry. The angle H is the apparent projection of SCK on a plane perpendicular to the plane of sight, the angle G is the apparent projection of $K\ C\ E$ on the same plane, the angle P is the position angle of the spot Σ from the north, E, counted positive eastward, so that $\chi = H + G + P$, the position angle from the sun's pole S. At the same time ρ is the angular distance of Σ from C, as measured at the center of the sun. The prime meridian is the central meridian at one adopted epoch, mean midnight, December 31, 1853 (Carrington); Greenwich mean noon, June, 13.72 (1887), Bigelow. The rotation periods of the sun change in latitude, from a maximum, 26.68 days synodic at the equator, to a minimum, about 30.00 days at the poles. Carrington's adopted period of rotation is applicable to latitudes = 12°, and

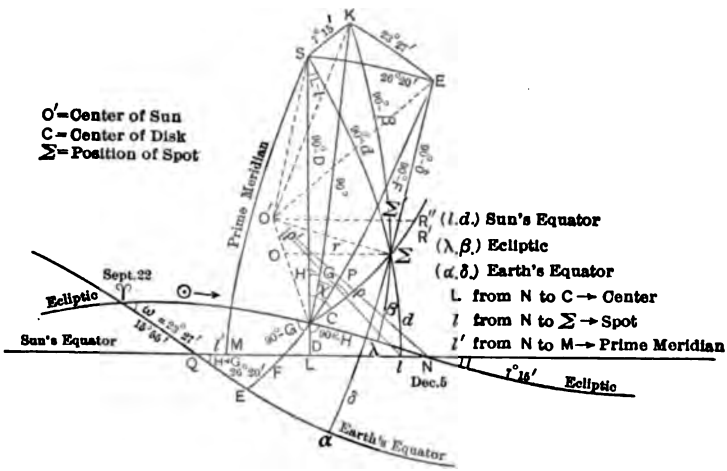


FIG. 72. Spherical positions on the sun's equator.

represents the average period of rotation for the sun-spots in that latitude alone. The radiation effects measured at the earth are for the equatorial period.

Poles. S = sun's equator; E = earth's equator; K = Ecliptic.

Inclinations. I = sun's equator to the ecliptic, 7°. 15′ = $KO'S$.
ω = earth's equator to the ecliptic, 23°.27′ = $E\,O'\,K$.

$H + G$ = earth's equator to the sun's equator, $26° \ 20' = E\,O'S$.

H = the projection of $S\,O'\,K$ at the center of the disk C.

G = the projection of $K\,O'\,E$ at the center of the disk C.

P = the position angle of $\Sigma = E\,C\,\Sigma$.

$\chi = H + G + P$.

The positive direction of the angles H, G, P, χ is through the east.

ρ = the heliocentric angle of Σ from $C = C\,O'\,\Sigma$.

Co-ordinates of the *center* of the disk C.

Sun's equator:
- L = longitude from the node N on the sun's equator, $N\,O'\,L$.
- D = latitude from the sun's equator = $C.\,L$.

Earth's equator:
- E = right ascension from the node Υ on the earth's equator, $\Upsilon\,O'\,E$.
- F = declination from the earth's equator = $C\,E$.

Ecliptic:
- $\odot$ = sun's celestial longitude from the node Υ on the ecliptic, $\Upsilon\,O'\,C$.
- ... = the latitude is zero = $C\,C$.

Co-ordinates of the *spot or point* Σ on the disk.

Sun's equator:
- l = longitude from the node N on the sun's equator, $N\,O'\,l$.
- d = latitude from the sun's equator = $l\,\Sigma$.

Earth's equator:
- α = right ascension from the node Υ on the earth's equator $\Upsilon\,O'\,\alpha$.
- δ = declination from the earth's equator, $\alpha\,\Sigma$.

Ecliptic:
- λ = heliocentric longitude from the node Υ on the ecliptic, $\Upsilon\,O'\,\lambda$.
- β = heliocentric latitude from the ecliptic = $\lambda\,\Sigma$.

l' = longitude of the prime meridian from N on sun's equator.

$L\text{-}l$ = heliocentric longitude from the central meridian, or difference of heliographic longitudes of center of disk and spot.

$L\text{-}l'$ = heliographic longitude of center of disk from the prime meridian.

$l\text{-}l'$ = heliographic longitude of Σ from the prime meridian.

Reduction of the photographic plate.

R = solar radius on the photograph.

R' = solar radius corrected for distortion on the plate.

R'' = solar radius as given in the ephemeris.

r = the perpendicular distance of Σ from the line $C\,O\,O'$, $= O\,\Sigma$.

r' = the measured distance of the spot from the center of the sun-picture, corrected for distortion.

ρ' = the angular distance $O\,\Sigma$ as seen from the earth.

(750) $\rho' = \dfrac{r'}{R'}R''$. $\qquad \dfrac{r'}{R'} = \sin O'\,\Sigma\,\Sigma' = \sin(\rho + \rho')$.

(751) $\rho + \rho' = \sin^{-1}\dfrac{r'}{R'}$. $\qquad \rho = \sin^{-1}\dfrac{r'}{R'} - \rho'$.

From the right spherical triangle ♈ $E\,C$ (Chauvenet, p. 172, 88),

(752) $\tan G = \tan\omega\cos\odot$; from $\cos\odot = \cot\omega\cot(90\text{-}G)$.

From the right spherical triangle $N\ L\ C$ (Chauvenet, p. 172, 88),

(753) $\tan H = \cos(\odot - N)\tan I$; from $\cos(\odot - N) = \cot I \cot(90°\text{-}H)$.

From the right spherical triangle $N\ L\ C$ (Chauvenet, p. 171, 86),

(754) $\sin D = \sin(\odot - N)\sin I$.

From the right spherical triangle $N\ L\ C$ (Chauvenet, p. 171, 87),

(755) $\tan L = \tan(\odot - N)\cos I$.

From the spherical triangle $S\ C\ \Sigma$, two sides and the included angle known, $90\text{-}D$, χ, ρ (Chauvenet, p. 179, M),

(756) $\sin d = \cos \rho \sin D + \sin \rho \cos D \cos \chi.$

From the spherical triangle $S\,C\,\Sigma$, from the two sides and angle opposite one of them (Chauvenet, p. 193, 148),

(757) $\text{Sin}\,(L - l) = \sin \chi \sin \rho \sec d.$

From the right spherical triangle ♈ $E\,C$ (Chauvenet, p. 171, 86, 87),

(758) $\sin F = \sin \odot \sin \omega.$

(759) $\tan E = \tan \odot \cos \omega.$

T = the fraction of a revolution executed by the prime meridian at a given date.

t = time from the epoch, June 13.72, 1887.

K = the mean angular velocity of the sun on its axis.

n = the mean angular velocity of the earth around the sun.

$K - n$ = the synodic angular velocity of the sun.

m = the complete number of sidereal rotations of the prime meridian M since the epoch.

(760) $$T = \frac{t}{24.863} - m \text{ (Bigelow).}$$

(761) $$l' = T \times \frac{360^\circ}{24.863} = T \times 14^\circ.4783 \text{ (Bigelow).}$$

To transform $(\lambda\,.\,\beta)$ to $(L\,.\,D)$, compute the auxiliary angle a,

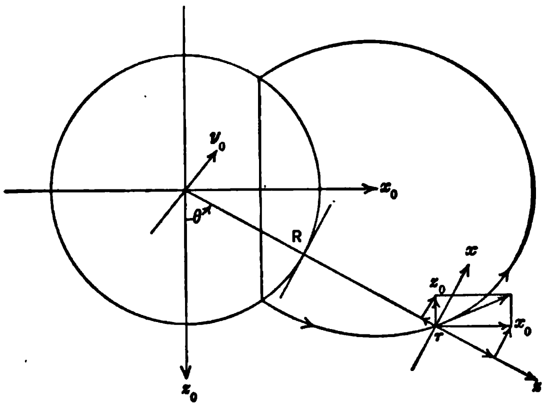
(762) $\tan a = \sin \lambda \cot \beta.$ Then,

(763) $$\tan L = \frac{\sin\,(I + a)}{\sin a} \tan \lambda.$$

(764) $$\sin D = \frac{\cos\,(I + a)}{\cos a} \sin \beta.$$

The Magnetic Fields of the Earth and the Sun

The sun presents many aspects of a magnetized sphere, with the positive pole on the south side of the sun's equator; the earth is a magnetized sphere with its positive pole on the south side of the equator. An isolated (supposed) positive (+) magnetic pole tends from the south side to the north side of the sun's equator, and from the south side to the north side of the earth's equator.



(x) Magnetic pole near the south geographical pole.

FIG. 73. Magnetic coordinates and component forces.

The following formulas are convenient for reference:

(765) $$\text{Mass} = M = \frac{4}{3}\pi R^3.$$

(766) $$\text{Mass Potential } P = \frac{M}{r} = \frac{4}{3}\pi R^3 . \frac{1}{r}$$

(767) $$\text{Vector Potential } V_e = -I\frac{dP}{dz} = \frac{4}{3}\pi I \frac{R^3}{r^3} z = \frac{4}{3}\pi I \frac{R^3}{r^2} \cos\theta \text{ (external).}$$

(768) $$V_i = \frac{4}{3}\pi I . z = \frac{4}{3}\pi I . r \cos\theta \text{ (internal).}$$

Exterior Forces

(769) $$Z_0 = -\frac{dV_e}{dz} = -\frac{4}{3}\pi I \frac{R^3}{r^3}\left(1 - \frac{3z^2}{r^2}\right) = -\frac{4}{3}\pi \frac{R^3}{r^3} I\,(1 - 3\cos^2\theta).$$

(770) $$X_0 = -\frac{dV_e}{dx} = +\frac{4}{3}\pi I \frac{R^3}{r^3} . 3\frac{zx}{r^2} = +\frac{4}{3}\pi \frac{R^3}{r^3} I . 3 \sin\theta \cos\theta.$$

(771) $Y_0 = -\frac{d V_e}{d y} = +\frac{4}{3}\pi I \frac{R^3}{r^3} \cdot \frac{3 z y}{r^2} = 0$. since $y = 0$.

(772) $Z = F_n = Z_o \cos\theta + X_o \sin\theta = \frac{4}{3}\pi I \frac{R^3}{r^3}$

$$(-\cos\theta + 3\cos^3\theta + 3\sin^2\theta\cos\theta)$$

$$= \frac{4}{3}\pi I \frac{R^3}{r^3} \cdot 2\cos\theta.$$

(773) $X = F_t = -Z_o \sin\theta + X_o \cos\theta = \frac{4}{3}\pi I \frac{R^3}{r^3}$

$$(\sin\theta - 3\cos^2\theta\sin\theta + 3\sin\theta\cos^2\theta)$$

$$= \frac{4}{3}\pi I \frac{R^3}{r^3} \cdot \sin\theta.$$

(774) $F^2 = X_o^2 + Z_o^2 = X_e^2 + Y_e^2 = \left[\frac{4}{3}\pi I \frac{R^3}{r^3}\right]^2 (1 + 3\cos^2\theta)$.

Surface $R = r$

(775) $Z_e = \frac{4}{3}\pi I \,.\, 2\cos\theta$

(776) $X_e = \frac{4}{3}\pi I \,.\, \sin\theta$

Pole $\begin{cases} Z_e = \frac{8}{3}\pi I. \\ X_e = 0. \end{cases}$

Equator $\begin{cases} Z_e = 0. \\ X_e = \frac{4}{3}\pi I. \end{cases}$

Interior Forces

(777) $Z_i = -\frac{d V_i}{d z} = -\frac{4}{3}\pi I.$

(778) $X_i = -\frac{d V_i}{d x} = 0.$

(779) $Y_i = -\frac{d V_i}{d y} = 0.$

(780) Line of Force $N = \frac{16}{3}\pi^2 R^3 I \cdot \frac{\sin\theta}{r} = \text{constant}.$

(781) Equipotential Surface $V = \frac{8}{3}\pi R^3 \frac{\cos\theta}{r^2} = \text{constant}.$

A magnetized sphere may be induced by two layers of positive and negative masses, circulating surface electric currents, two positive and negative poles placed near the center, and by other physical devices, as rotating Ampere electric currents in the interior around lines parallel to the axis, in which case the sphere is polarized. This is probably the case for the earth and for the sun. For surface exterior currents, as in the earth's atmosphere, it is evident that variation in the east-west direction changes the strength of the existing magnetic field along the lines of force. The vectors of Fig. 68, 69, can only be produced by north and south ionization currents, as there indicated, distinct from east and west currents on the normal field. The asymetric position of the sun's magnetic poles deduced by Bigelow in 1891 has been confirmed by direct observation at the Mt. Wilson observatory, 1913.

Conclusion

The general functional relations between the incoming solar radiation, the portion of it transformed into ionization currents, the magnetic disturbing vectors depending upon them, all remain to be discovered. It has been possible to indicate in this Treatise some of the important elements in these problems, and it is hoped that the formulas and methods of discussion here adopted will greatly facilitate the pursuit of such researches by many students. The practical side of the matter consists in the development of the branch of Meteorology and Solar Physics which will culminate in the ability to predict the seasonal climatic conditions likely to prevail during the coming year in the several large agricultural regions of the earth. The extent and scope of these subjects are so great that the co-operation of many institutions and national offices will be essential for the success of so important an enterprise.

CHAPTER VII

Extension of the Thermodynamic Computations to the Top of the Atmosphere

Remarks on the Bouguer Formula

THE computations on the thermodynamic data of the earth's atmosphere have heretofore in the examples been limited to about 20,000 meters, but it is very desirable to extend them to the vanishing plane of the gaseous envelope. The following summary illustrates a method for accomplishing this purpose, and the results afford approximate material for further important researches. These concern the problems of the solar radiation and its absorption, together with the correlative terrestrial radiation and its absorption. It is especially proposed to test the result that the "solar constant," as derived from pyrheliometer observations is about 1.92 $\frac{gr.\ cal.}{cm.^2\ min.}$, or whether it is about twice as much, in conformity with the bolometer data. The Bouguer formula of reduction, in the preceding notation,

$$I = I_0\, p^m = I_0\, p^{m_0 \cdot \frac{m}{m_0}} = I_0\, p^{m_0 \sec \zeta}$$

has been shown to contain two unknown terms I_0, m_0, which together determine the source of the effective radiation, so soon as their details are understood. It has been customary to identify I_0 with the solar radiation falling on the outer limit of the atmosphere for which it is assumed that $m_0 = 1$ and disappears from the equation, enabling it to be solved for I_0. This is, in fact, a special interpretation by assumption, and doubt is thrown upon it by the bolometer ordinates, since these require an energy spectrum curve of about $T = 6{,}900°$ (3.88 calories), while $I_0 = 1.92$ calories is satisfied with 5,800° for the effective solar temperature. It has, furthermore, been shown by the pyrheliometer observations that I_0 is not a constant quantity, that is, a source of radiation independent of atmospheric conditions as to aqueous vapor, dust and ice contents, not now referring to the small variations of the radiation energy emitted at the sun itself. The evidence is that as the coefficient of transmission p diminishes in value the term I_0 is progressively depleted. The extrapolated points I_0 on the axis of ordinates do not concentrate at $I_0 = 1.92$ calories. It follows that I_0 is a terrestrial variable, and it is the purpose of this chapter to investigate further the properties of the terms, I_0, m_0.

I. First Distribution of Temperature

The balloon ascension, Huron, So. Dak., September 1, 1910

The balloon ascension, Huron, So. Dak., September 1, 1910, reached a height of 30,000 meters, recording temperatures T, but without velocities q. These have been provisionally supplied by taking the mean values for ten other ascensions in order to balance approximately the terms of the gravity equation,

$$(196) \quad g\,(z_1 - z_0) = -\frac{P_1 - P_0}{\rho_{10}} - \tfrac{1}{2}\,(q_1^2 - q_0^2) - (Q_1 - Q_0).$$

Table 82 contains the adopted T, q, from which all the other terms were computed up to 30,000 meters, and the results were plotted on large diagrams of which Figs. 74, 75, are reduced copies. The method of extending the T data to the higher planes is as follows: On the 30,000-meter plane $T = 232°.3$, $P = 1352$ ($B = 10.14$ mm. as computed, while the automatic register recorded $B = 10.70$ mm.), $\rho = 0.05512$, $R = 105.61$, the check equation $P = \rho R T$ being perfectly satisfied. At first the P and ρ curves were extended graphically, and from these values up to 40,000 meters approximate values of T were computed, carrying the T curve around the corner of the isothermal region into a very rapid gradient. These temperatures are necessary to continue the P, ρ, curvatures at that elevation. Then all the other terms were computed directly from T and the diagrams were extended. At 40,000 meters $T = 170°.5$, $P =$

TABLE 82

SUMMARY OF THE THERMODYNAMIC DATA COMPUTED FROM THE BALLOON ASCENSION TO 30,000 METERS MADE AT HURON, SOUTH DAKOTA, SEPTEMBER 1, 1910

z	T	P	ρ	C_p	q	W_1-W_0	U_1-U_0	K_{10}	log c	a
Height, in Meters	Temperature	Pressure	Density	Specific Heat Ratio	Velocity m. p. s.	External Work of Expansion	Inner Kinetic Energy	Radiation	Coefficient in the Law	Exponent in the Law
50000	0.0	....	....	7.86	...	−174.	−9033.	....	−5.074	3.600
49	2.6	4.18×10^{-6}	4.900×10^{-8}	113.66	...	+808.	−7600.	− .0003	−5.295	3.700
48	6.3	1.48×10^{-2}	1.637×10^{-5}	497.19	...	2002.	−5370.	− .1156	−5.448	3.770
47	11.9	0.717	2.586×10^{-4}	807.42	...	3379.	−4615.	− 1.501	−5.585	3.830
46	19.6	6.56	.001247	929.28	...	6204.9	−6689.1	− 13.063	−5.738	3.905
45000	28.7	27.47	.003452	959.75	...	5616.5	−7042.3	− 51.36	−5.618	3.843
44	48.5	67.69	.006555	738.48	...	4321.4	−7830.8	− 158.0	−5.427	3.760
43	78.0	117.63	.009710	537.55	...	3245.2	−8281.1	− 348.2	−5.274	3.691
42	108.7	170.2	.01263	429.16	...	2731.6	−8651.1	− 614.5	−5.161	3.640
41	140.9	224.	.01539	358.49	...	2336.3	−8815.8	− 934.6	−5.077	3.602
40000	170.5	279.	.01796	315.71	...	2141.7	−8931.6	−1278.2	−5.025	3.578
39	192.3	337.	.02054	295.59	...	2016.1	−8948.6	−1607.9	−4.999	3.567
38	208.2	400.	.02373	286.81	...	1993.1	−8955.7	−1919.4	−4.992	3.564
37	216.1	470.	.02600	289.48	...	2036.8	−8960.1	−2218.7	−5.000	3.568
36	222.0	549.	.02905	294.77	0.0	2108.2	−8958.0	−2522.	−5.011	3.573
35000	224.8	640.	.03239	304.28	1.0	2153.0	−8894.8	−2830.	−5.028	3.580
34	225.8	744.	.03607	316.45	2.0	2281.7	−8904.9	−3155.	−5.046	3.588
33	227.4	866.	.04015	328.22	2.8	2324.7	−8832.4	−3517.	−5.066	3.596
32	229.2	1005.	.04466	340.02	3.2	2428.0	−8818.5	−3926.	−5.081	3.604
31	231.1	1166.	.04963	352.02	3.6	2527.8	−8792.4	−4388.	−5.095	3.610
30000	232.3	1352.	.05512	365.58	3.8	2599.2	−8722.9	−4849.	−5.120	3.622
29	232.5	1565.	.06118	380.93	3.5	2748.1	−8718.2	−5357.	−5.140	3.630

28	232.9	1814.	.06795	396.82	3.0	2809.0	−8619.8	−5960.	−5.157	3.638
27	233.3	2098.	.07536	413.15	2.5	2955.5	−8579.9	−6539.	−5.181	3.649
26	232.0	2429.	.08363	433.41	3.0	3128.3	−8546.5	−7155.	−5.201	3.658
25000	230.4	2817.	.09290	455.48	3.5	3258.1	−8455.6	−7807.	−5.226	3.670
24	229.1	3267.	.10320	478.10	4.0	3430.5	−8403.4	−8656.	−5.252	3.681
23	228.3	3794.	.1148	500.94	4.5	3524.0	−8267.0	−9337.	−5.270	3.689
22	227.7	4411.	.1278	524.64	5.0	3781.7	−8255.8	−10263.	−5.295	3.700
21	224.7	5130.	.1423	555.35	5.5	3986.1	−8155.0	−11295.	−5.323	3.713
20000	222.6	5975.	.1586	585.83	6.5	4229.2	−8107.5	−12366.	−5.344	3.723
19	221.8	6972.	.1770	614.77	7.0	4400.0	−7986.7	−13643.	−5.375	3.737
18	220.9	8132.	.1974	645.31	7.2	4653.1	−7908.8	−14931.	−5.398	3.747
17	220.0	9500.	.2205	677.75	9.8	4832.6	−7711.4	−16248.	−5.417	3.755
16	218.9	11095.	.2463	712.38	15.2	5156.9	−7724.3	−17972.	−5.442	3.767
15000	218.2	12986.	.2754	747.92	17.2	5339.8	−7549.7	−19764.	−5.461	3.775
14	219.0	15183.	.3078	779.60	20.3	5559.4	−7428.8	−21817.	−5.482	3.784
13	218.5	17742.	.3439	817.34	22.9	5851.8	−7489.2	−24012.	−5.504	3.795
12	220.9	20731.	.3841	845.63	21.2	5998.7	−7424.6	−27704.	−5.514	3.799
11	226.5	24151.	.4282	861.92	18.9	6096.6	−7351.7	−31391.	−5.517	3.800
10000	233.0	28024.	.4760	874.46	17.9	6172.5	−7317.5	−35695.	−5.525	3.804
9	240.1	32376.	.5274	884.96	16.7	6246.0	−7285.2	−40495.	−5.523	3.803
8	247.5	37249.	.5827	893.96	16.0	6315.2	−7283.7	−46012.	−5.526	3.805
7	254.4	42679.	.6419	903.88	13.7	6391.6	−7237.1	−51620.	−5.532	3.807
6	260.8	48729.	.7054	916.86	11.8	6441.1	−7205.5	−58156.	−5.535	3.808
5000	270.0	55421.	.7730	919.18	10.1	6472.7	−7193.2	−65570.	−5.534	3.808
4	278.4	62779.	.8446	924.12	9.0	6522.4	−7164.9	−73187.	−5.531	3.806
3	285.1	70873.	.9207	934.58	8.0	3337.9	−3589.0	−78707.	−5.538	3.810
2500	284.8	75278.	.9611	952.03	7.6	3369.9	−3532.6	−81358.	−5.547	3.814
2000	284.4	79932.	1.0029	970.04	7.3	3427.9	−3522.1	−84746.	−5.548	3.815
1500	286.1	84868.	1.0465	981.12	6.5	3457.6	−3505.6	−88614.	−5.554	3.817
1000	288.8	90064.	1.0917	988.78	5.6	3437.4	−3453.2	−92283.	−5.555	3.818
500	292.6	95508.	1.1385	992.66	5.3	754.8	−756.9	−96298.	−5.556	3.818
392	293.4	96723.	1.1485	993.58	5.0	......	......	......	−5.557	3.819

279 (B = 0.002095 mm.), ρ = 0.01796, R = 91.20, and $P = \rho R T$ is again exactly satisfied. These data indicate that the vanishing plane of the atmosphere is approaching, and a trial was made of z = 50,000 meters for T = 0° the approximate temperature of space. Further experience with the data indicates that

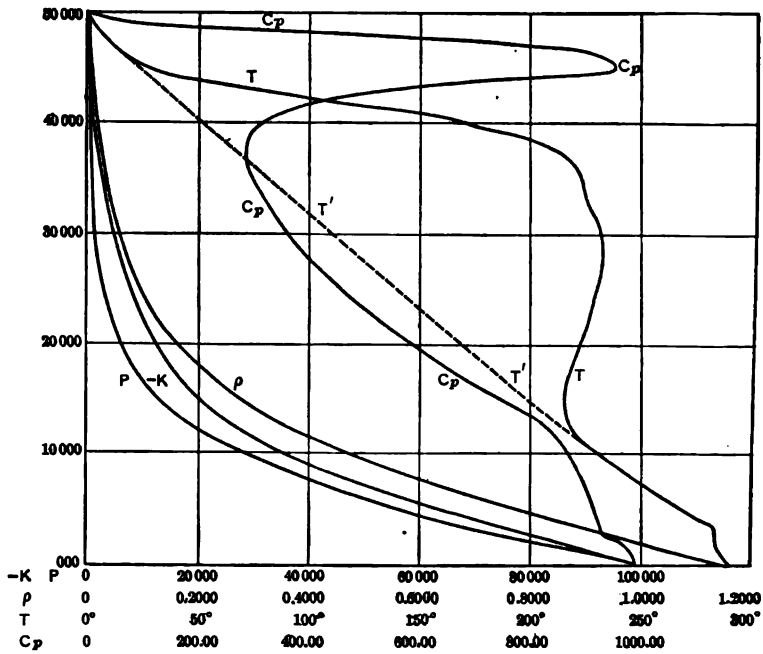


FIG. 74. The dynamic data to the top of the atmosphere at 50,000 meters, from the balloon ascension, Huron, S. D., Sept. 1, 1910

P = Pressure in kilograms per square meter.
$-K$ = Radiation energy.
ρ = Density in kilograms per cubic meter.
T = Temperature in absolute degrees.
T^1 = Temperature before the formation of the isothermal region.
Cp = Specific heat ratio at constant pressure.

The balloon ascension, Huron, S. D., September 1, 1910, reached the elevation 30,000 meters. The lines for $P, -K, \rho, Cp$, and T (up to the isothermal level) converge at a point on the vanishing plane 50,000 meters. By a series of trials T was determined such as to make the $P, -K, \rho$ curves run out smoothly to a common vanishing point.

the vanishing plane is nearer 90,000 meters, but the example is reproduced because several interesting points are illustrated. The temperature T at any level, ***whether observed or adopted***, can always be fully checked by the combination of these two equations. If there is an important residual for $g(z_1 - z_0)$ it must be due to the fact that an erroneous T was ascribed to the level. In this case the residuals are large—from 41,000 to 50,000 meters, but this can be remedied by further trial computations in

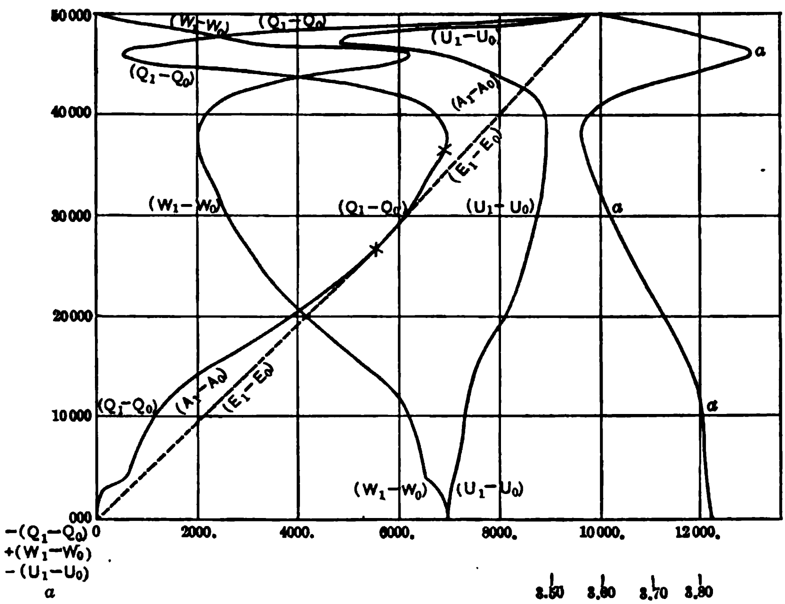


FIG. 75. The thermodynamic and radiation data to the top of the atmosphere at 50,000 meters, from the balloon ascension, Huron, S. D., Sept. 1, 1910.

(Q_1-Q_0) = The free heat transmitted by the solar radiation.
(W_1-W_0) = The external work of expansion.
(U_1-U_0) = The inner energy of molecular motion.
a = The exponent in the radiation equation.
(E_1-E_0) = The total incoming free heat.
(A_1-A_0) = The absorbed portion of it.

$$(Q_1-Q_0) = (W_1-W_0) + (U_1-U_0)$$
$$E_1-E_0 = (Q_1-Q_0) + (A_1-A_0)$$

connection with other high-level balloon ascensions. It should be noted in this example that the value of $R = 268.46$ at 46,000 meters has resumed nearly its surface value, 287.03. This means that the thermodynamic system has been subjected to great heating influences in the levels above 40,000 meters, and this is easily seen by studying the data of Table 82, and Figs. 74, 75. The P, $-K$, ρ, curves run out smoothly to the vanishing plane, but the $Cp = R\frac{k}{k-1}$, $(Q_1 - Q_0)$, $(W_1 - W_0)$, $(U_1 - U_0)$, log c, a, curves all are much distorted in the strata above 40,000 meters. This is the thermodynamic result of heating the rarefied gases of the outer levels from $T = 0°$ on the vanishing plane to $T = 208°.2$ at 38,000 meters. The so-called isothermal layer from 11,000 to 38,000 meters is merely that part of the atmosphere where the surviving solar radiation nearly balances the terrestrial radiation. Above 38,000 meters is a region of very powerful absorption of the solar radiation, in which the energy of the short waves of the solar spectrum, $\lambda = 0.00\ \mu$ to $0.40\ \mu$, are absorbed, the energy being used in heating this stratum through about 208 degrees. This is the true albedo of the earth's atmosphere, and it amounts to nearly one-half of the total solar radiation falling on the earth's outermost stratum of atmosphere. If a line be drawn, T^1, from the lower part to the upper part of the T-curve, the enclosed area between T and T^1 represents the region of absorption, and the isothermal curve to the right of it is one of its three boundaries. This area can be studied for several thermodynamic conditions.

The divergence between the pressure curve P and the radiation energy curve $-K$ is due to the free heat per volume $(Q_1 - Q_0) = (W_1 - W_0) + (U_1 - U_0)$, where $(Q_1 - Q_0)$ represents the excess of the inner energy $(U_1 - U_0)$ over the work of expansion $(W_1 - W_0)$. At the points where the P and $-K$ curves coincide, as at the surface, there is no free heat at liberty to move. In the case of an adiabatic atmosphere, $(Q_1 - Q_0) = 0$, and there is no radiation. It should be carefully noted that meteorological discussions and the tables in common use have been heretofore uniformly based upon strictly adiabatic formulas. Since the adiabatic and non-adiabatic systems are identical at the surface, this distinction is not important there, but because these systems diverge to a large amount in the upper atmosphere the inference follows that all meteorological tables based upon R = constant produce erroneous values of the density above the surface. For example, with R = constant, $g\ (z_1 - z_0) = - Cp_a\ (T_a - T_0) = - \frac{P_1 - P_0}{\rho_{a0}}$, and from this comes the adiabatic gradient $\frac{g}{Cp_a} = - \frac{(T_a - T_0)}{(z_1 - z_0)} = -9.870$ per 1,000 meters. If, on the other hand, the terms for circulation $- \frac{1}{2}\ (q_1^2 - q_0^2)$ and for radiation $- (Q_1 - Q_0)$ are to be introduced, this can only be done by changing ρ_{a0} to ρ_{10}, since the pressure term $(P_1 - P_0)$ is fixed by direct measures at heights which do not correspond with any adiabatic temperature gradient. It follows that we have another value of the specific heat ratio Cp_{10} which differs from Cp_a, so that $- Cp_{10}\ (T_a - T_0) = - \frac{P_1 - P_0}{\rho_{10}}$ and the non-adiabatic ρ_{10} differs from the adiabatic ρ_a. By subtraction we find that,

$$(198)\quad - (Cp_a - Cp_{10})\ (T_a - T_0) = - \tfrac{1}{2}\,(q_1^2 - q_0^2) - (Q_1 - Q_0),$$

as heretofore indicated. If for the moment we assume that $- \frac{1}{2}\ (q_1^2 - q_0^2) = 0$, as is the case at the top and at the bottom of the atmosphere, there are two extreme values of $(Q_1 - Q_0)$:

1. At the vanishing plane, for $Cp_{10} = 0$,

$$- (Q_1 - Q_0) = - (Cp_a - Cp_{10})\quad (T_a - T_0) = 993.58 \times 9.87$$
$$\doteq 9806, \text{ per } 1{,}000 \text{ m.}$$

2. At the surface, for $Cp_{10} = Cp_a$,

$$- (Q_1 - Q_0) = - (Cp_a - Cp_{10})\ (T_a - T_0) = 0.$$

The adiabatic formula requires that $Cp_{10} = Cp_a$ from the surface to the vanishing plane, which is in disagreement with the observed temperature gradients.

Illustrations of the Use of Erroneous Densities

In order to illustrate the amount of divergence between ρ_a and ρ, we have for this balloon ascension—

TABLE 83

THE DIFFERENCE BETWEEN ρ (NON-ADIABATIC) AND ρ_a (ADIABATIC) DENSITY

z	ρ	ρ_a
50000	0.0000	0.0000
45000	0.0035	0.0033
40000	0.0180	0.0057
35000	0.0324	0.0099
30000	0.0551	0.0203
25000	0.0929	0.0426
20000	0.1586	0.0935
15000	0.2754	0.2073
10000	0.4760	0.4190
5000	0.7730	0.7151
surface	1.1485	1.1485

Since ρ is greater than ρ_a it follows that,

(782) $$-\frac{P_1 - P_0}{\rho_{10}} \text{ is smaller than } -\frac{P_1 - P_0}{\rho_{a0}}$$

and this is compensated in the gravity equation by the terms for the circulation and radiation. It has been tested by numerous computations that $g\,(z_1 - z_0) = -\,Cp_a\,(T_a - T_0) = -\,\frac{P_1 - P_0}{\rho_{a0}}$ without important residuals from the surface to the vanishing plane, and that $-\,Cp_{10}\,(T_a - T_0) = -\,\frac{P_1 - P_0}{\rho_{10}}$ holds true throughout the atmosphere. No system of meteorology founded on $R_a \frac{k}{k-1} = Cp_a =$ constant is applicable in practical studies of atmospheric conditions.

There is another important matter in which erroneous densities have been introduced into studies of radiation phenomena.

It has been thought proper to modify the Bouguer formula in such a way that the observed variations in I, p, shall be dependent upon the nature of the gaseous path m, so that the integrated amount of aqueous vapor in the form of liquid water may be made a factor attaching to the exponent m. Thus it has been computed that the water mass $\Sigma\mu = 2e_0$, where $\Sigma\mu$ is the integral of the water contents in grammes and e_0 the vapor pressure in millimeters of mercury at the surface. The hair hygrometer measures the relative humidity of the air at the different levels, and the corresponding e in mm. is computed. The full formula is easily shown to be,

$$(783) \qquad \mu = \rho_0 \frac{273}{T} \cdot \frac{287}{R} \cdot \frac{B}{760} \left(0.622 \frac{e}{B} + 0.235 \frac{e^2}{B^2}\right).$$

It has been customary to assume $R = 287$ and to omit the term in $\frac{e^2}{B^2}$, so that there remains,

$$(784) \qquad \mu = \rho_0 \frac{273}{T} \times 0.622 \frac{e}{760}.$$

This formula diverges from the true one in the same way that ρ differs from ρ_a at the different levels, so that the sum of μ, $\Sigma\mu$, is erroneous in the formula $\Sigma\mu = 2e$, being too small.

Furthermore, the solar radiation suffers two sorts of depletion, the first as true absorption upon which p depends, the second as scattering upon which I_0 partly depends. We have shown how I_0 should be modified for vapor contents $(I_0 - 0.0214\, e)$ before introducing it into the Bouguer Formula. The ice crystals of the upper strata, 8,000 to 25,000 meters, are effective reflectors of the radiation energy in these strata. The hair hygrometer takes no account of the ice contents in the (∂-stage), nor the water contents (γ and β stages) which may be present, but only of the vapor contents (α-stage). It follows that since water and ice exist throughout the lower atmosphere up to 25,000 meters, but not always in cloud forms, though these have been observed up to 16,000 meters, that the α-stage Formula is inadequate to the purpose which has been imposed upon it.

The remedy for these difficulties is being considered in a special research.

The Thermodynamic Terms

Table 82 and Fig. 75 give the relations of $(Q_1 - Q_0) = (W_1 - W_0) + (U_1 - U_0)$. $(Q_1 - Q_0)$ begins at the surface without value because the external work and the inner energy are in equilibrium; it increases by a curve determining the amount of the lower absorption to a point between the elevations 26,000 and 27,000 meters, this being the level where $2\ (Q_1 - Q_0) = (U_1 - U_0) - (W_1 - W_0)$, which defines the true isothermal level where there is no absorption of solar energy, in the same sense that there is no absorption at the top and at the bottom of the atmosphere. This gives three points on the line of total solar energy, 9,806 on the vanishing plane, about 5,780 at 26,800 meters, and 0 at the surface. The $(E_1 - E_0)$ line of total solar radiation energy on the several levels was drawn by connecting these three points. The area between the axis of ordinates and the $(Q_1 - Q_0)$ curve represents the free heat of transmission, the area between the $(Q_1 - Q_0)$ and the $(E_1 - E_0)$ curves that of absorption, while $(E_1 - E_0) = (Q_1 - Q_0) + (A_1 - A_0)$, the total area. The data on the several levels are for 1,000-meter areas, except below 3,000 meters, where the vertical height is 500 meters. The sum of these areas from the surface to 50,000 meters apparently represents the true "solar constant" of radiation energy, with the constituents of transmission and absorption.

The $(Q_1 - Q_0)$ curve continues to increase in value upward, as $(W_1 - W_0)$ and $(U_1 - U_0)$ separate, till a maximum value for the free heat of transmission is reached at $-$ 6963 on the 38,000-meter level. It will be shown that the pyrheliometer receives an amount of heat at the surface such as results from a summation up to about this elevation, as if this were the general efficient source of radiation at about 1.92 calories. The $(Q_1 - Q_0)$ and $(E_1 - E_0)$ curves begin to diverge at 27,000 meters, and very rapidly after passing 38,000 meters. $(Q_1 - Q_0)$ falls to a minimum at 48,000°, and rises immediately to its primary maximum, 9,806 at 50,000 meters, in this computation. $(W_1 - W_0)$ and $(U_1 - U_0)$ pass through entirely similar changes above 38,000 meters, and they all correspond to the very rapid increase in the temperature of the gases of the upper absorption level. It is evident that the pyrheliometer measures the transmission and the minor absorption up to about 38,000 meters, but is not cognizant of the great absorption area above it where the absorption is nearly complete. The bolometer measures the relative ordinates of the true solar spectrum, unless some of them are absorbed, but for its total energy depends upon the computations and integrations of the spectrum curve at given temperatures. This is shown by the spectrum to have lost nearly half the incoming energy in the short waves having long ordinates, these corresponding with the upper absorption region, besides some minor absorptions attributed to aqueous vapor selective absorption bands belonging to the lower region. The general slope of the $(E_1 - E_0)$ line indicates that the equal terrestrial radiation is required to balance the solar radiation, generally large below but small above, in accordance with the prevailing temperatures and the equation of equilibrium, as will be explained.

The corresponding values of K_{10}, log c, a, in $K_{10} = c\ T^a$, have been computed. The exponent $a = 3.819$ at the surface gradually falls to a minimum 3.564 at 38,000 meters; it rises to a maximum 3.905 at 45,000 meters, then falls to about 3,600 at the vanishing plane. Further computations will be undertaken in respect to the region above 38,000 meters. (Fig. 79.)

The Constituents of the Solar and the Terrestrial Radiations in the Earth's Atmosphere

It will be convenient to transform the thermal data which have been computed in the (Meter, Kilogram-second) (M. K. S.) system into the corresponding values in the (Centimeter-gram-second) (C. G. S.) system, before proceeding to a further discussion of the transmitted and absorbed constituents of the solar and the terrestrial radiations. The term $(Q_1 - Q_0)$ in

$$g_0 (z_1 - z_0) = - Cp_a (T_a - T_0) = - \frac{P_1 - P_0}{\rho_{10}} - \tfrac{1}{2} (q_1^2 - q_0^2) - (Q_1 - Q_0)$$

must satisfy the dimensions of all these terms, as well as the equations $P = \rho R T$, and $(Q_1 - Q_0) = (W_1 - W_0) + (U_1 - U_0)$. $(Q_1 - Q_0)$ was derived from $- (Cp_a - Cp_{10}) (T_a - T_0) + \frac{1}{2} (q_1^2 - q_0^2) = - (Q_1 - Q_0)$ so that it has the same dimensions as $R = \frac{k-1}{k} Cp$, and since $R = \frac{P}{\rho T} = \frac{Pv}{T}$ they are the same as those for work, $(W_1 - W_0) = P_{10} (v_1 - v_0)$, and for the inner energy $(U_1 - U_0) = Cv (T_a - T_0)$. In transformations from the (M. K. S.) to the (C. G. S.)system the dimension factors are, $[M] = 1000 = 10^3$, $[L] = 100 = 10^2$; in transformations from the (C. G. S.) to the (M. K. S.) system, $[M] = 10^{-3}$, $[L] = 10^{-2}$. All are in mechanical units.

(4) Gravity in meters/sec, $[g_0] = L$; $(z_1 - z_0)$ meters, $[z_1 - z_0] = L$ } $g_0 (z_1 - z_0) = L^2 = 10^4 = \ldots\ldots = 10000$

Pressure, kilog./met.², $[P_1 - P_0] = ML^{-1}$; Density, kilog./met.³, $[\rho_{10}] = ML^{-3}$ } $\left[\frac{P_1 - P_0}{\rho_{10}}\right] = L^2 = 10^4 = \ldots = 10000$

Velocity square, $[q_1^2 - q_0^2] = L^2 = 10^4 = \ldots\ldots = 10000$

Specific heat, $[Cp, Cv, R. K.] = L^2 = 10^4 = \ldots\ldots = 10000$

Heat, $[Q_1 - Q_0 = (Cp_a - Cp_{10}) (T_a - T_0)] = L^2 = 10^4 = \ldots\ldots = 10000$

Work, $[(W_1 - W_0) = \bar{P}_{10} (v_1 - v_0)] = ML^{-1}.\ M^{-1} L^3 = L^2 = 10^4 = . = 10000$

Inner Energy, $[(U_1 - U_0) = Cv (T_a - T_0)] = L^2 = 10^4 = \ldots\ldots = 10000$

The common factor for reducing all the terms of the gravity and heat equations from (M. K. S.) to (C. G. S.) in mechanical units is $10^4 = 10{,}000$.

From the general equations $(W_1 - W_0) = P_{10} (v_1 - v_0)$ and $(Q_1 - Q_0) = (W_1 - W_0) + (U_1 - U_0)$, we have,

$$K_{10} = \frac{U_1 - U_0}{v_1 - v_0} = \frac{Q_1 - Q_0}{v_1 - v_0} - P_{10} = c\,T^a$$

which is the radiation equation employed in the computations. The dimensions of all these terms are the same in reducing from the (M. K. S.) to the (C. G. S.), the factor being $\frac{10000}{1000} = 10$, as is commonly used in the pressure P_{10}. The reduction from mechanical units to heat units (C. G. S.) is by the factor $\frac{1}{4.1851 \times 10^7}$, and the factor for change from second to minute is 60. Hence, to reduce $(Q_1 - Q_0)$ in M. K. S., mechanical units into $\frac{gram\ calorie}{cm.^2\ minute}$, the factor is,

$$\frac{10000 \times 60}{1000 \times 4.1851 \times 10^7} = \frac{6 \times 10^{-5}}{4.1851} = 0.000014336; \text{ Log.} - 5.15644.$$

Compare the constants, coefficients, and dimensions of Table 95, where the formulas afford many other combinations in conformity with the kinetic theory of gases.

The Total, Transmitted and Absorbed Amounts of the Solar and the Terrestrial Radiations

Huron, September 1, 1910

The method of separating the total radiation $(E_1 - E_0)$ received at the earth from the sun into the transmitted $(Q_1 - Q_0)$ and the absorbed $(A_1 - A_0)$ constituents on each level has been explained. The coefficient of transmission is $p = \frac{Q_1 - Q_0}{E_1 - E_0}$, and of absorption is $k = \frac{A_1 - A_0}{E_1 - E_0}$. The terrestrial radiation is obtained as follows: Let $J_0 = c\,T^4$, the black body radiation of the atmosphere at the point whose temperature is T; let $J_a = c\,T^a$ the actual radiation for c and a as computed from the thermodynamic data, so that the actual absorption between two planes is $J_{a \cdot 1} - J_{a \cdot 0} = c_1\ T_1^{a_1} - c_0\,T_0^{a_0}$. The general equation of radiation equilibrium is

$2\,J_0 = E + D$, where D is the total terrestrial radiation. Hence, $D = 2\,J_0 - E$, and $B = D - J_a$, where B is the transmitted terrestrial radiation. The coefficient of terrestrial transmission is $p^1 = \frac{B_1 - B_0}{D_1 - D_0}$, and that of absorption is $k^1 = \frac{J_{a \cdot 1} - J_{a \cdot 0}}{D_1 - D_0}$. The results of these computations appear on Table

TABLE 84

THE ATMOSPHERIC THERMAL QUANTITIES ATTRIBUTABLE TO SOLAR, ATMOSPHERIC AND TERRESTRIAL RADIATIONS, $\frac{gr.\ cal.}{cm.^2\ min.}$

BALLOON ASCENSION, HURON, SO. DAK., SEPTEMBER 1, 1910

z Height, in meters	Solar Radiation					Atmospheric (black)	Terrestrial Radiation					$Q_1 - Q_0$ in M.K.S. Mechanical Units
	Amount			Coefficient			Amount			Coefficient		
	$E_1 - E_0$ Total	$Q_1 - Q_0$ Trans.	$A_1 - A_0$ Absorb.	p Trans.	k Absorp.	$J_{a1} - J_{a0}$ (1) $c_1T^4 - c_0T^4$	$D_1 - D_0$ (3) Total	$B_1 - B_0$ (4) Trans.	$J_{a1} - J_{a0}$ (2) Absorb.	p^1 Trans.	k^1 Absorb.	
50000	−.1406	−.1320	−.0086	0.939	0.061	0.0000	+.1406	+.1406	.0000	1.000	0.000	−9207.
49	−.1380	−.0974	−.0406	.706	.294	.0000	+.1380	+.1380	.0000	1.000	.000	−6792.
48	−.1354	−.0483	−.0871	.357	.643	.0000	+.1353	+.1353	.0000	1.000	.000	−3368.
47	−.1328	−.0177	−.1151	.134	.866	−.0001	+.1325	+.1326	−.0001	1.000	.000	−1236.
46	−.1301	−.0069	−.1232	.053	.947	−.0003	+.1296	+.1298	−.0002	1.000	.000	− 484.
45000	−.1275	−.0204	−.1071	.160	.840	−.0017	+.1241	+.1247	−.0006	.999	−.001	−1426.
44	−.1249	−.0503	−.0746	.403	.597	−.0079	+.1092	+.1110	−.0018	.984	−.016	−3509.
43	−.1223	−.0722	−.0501	.590	.410	−.0190	+.0843	+.0870	−.0027	.969	−.031	−5036.
42	−.1197	−.0849	−.0348	.709	.291	−.0384	+.0428	+.0469	−.0041	.913	−.059	−5920.
41	−.1171	−.0929	−.0242	.793	.207	−.0608	−.0045	+.0018	−.0063	−.400	−.600	−6480.
40000	−.1145	−.0973	−.0172	.850	.150	−.0672	−.0199	−.0156	−.0043	.784	.216	−6790.
39	−.1119	−.0994	−.0125	.889	.111	−.0688	−.0257	−.0200	−.0057	.778	.222	−6933.
38	−.1092	−.0998	−.0094	.914	.086	−.0481	+.0130	+.0178	−.0049	.730	−.270	−6963.
37	−.1066	−.0993	−.0074	.931	.069	−.0445	+.0177	+.0226	−.0049	.782	−.218	−6923.
36	−.1040	−.0982	−.0058	.944	.056	−.0333	+.0373	+.0419	−.0046	.890	−.110	−6850.
35000	−.1014	−.0967	−.0048	.953	.047	−.0238	+.0539	+.0581	−.0043	.928	−.072	−6742.
34	−.0988	−.0950	−.0038	.961	.039	−.0326	+.0336	+.0390	−.0054	.861	−.139	−6623.
33	−.0962	−.0933	−.0029	.970	.030	−.0298	+.0366	+.0422	−.0056	.867	−.133	−6508.
32	−.0936	−.0916	−.0020	.979	.021	−.0321	+.0293	+.0348	−.0055	.842	−.158	−6391.
31	−.0910	−.0898	−.0012	.987	.013	−.0405	+.0101	+.0193	−.0092	.523	−.477	−6265.
30000	−.0884	−.0878	−.0006	.994	.006	−.0288	+.0307	+.0376	−.0069	.816	−.184	−6124.
29	−.0857	−.0856	−.0001	.998	.002	−.0272	+.0313	+.0395	−.0082	.792	−.208	−5970.
28	−.0831	−.0833	+.0002	1.000	.000	−.0373	+.0086	+.0194	−.0108	.443	−.557	−5811.

27	−.0805	−.0806	+.0001	1.000	.000	−.0170	+.0466	+.0540	−.0074	.863	−.137	−5624.
26	−.0775	−.0777	+.0002	1.000	.000	−.0200	+.0376	+.0481	−.0106	.782	−.218	−5418.
25000	−.0745	−.0745	.0000	1.000	.000	−.0258	+.0229	+.0347	−.0117	.660	−.340	−5198.
24	−.0715	−.0713	−.0002	.997	.003	−.0203	+.0310	+.0392	−.0083	.791	−.209	−4973.
23	−.0686	−.0680	−.0006	.992	.008	−.0342	+.0002	+.0153	−.0151	....	......	−4743.
22	−.0656	−.0641	−.0015	.978	.022	−.0087	+.0481	+.0616	−.0135	.781	−.218	−4474.
21	−.0626	−.0598	−.0028	.955	.045	−.0083	+.0458	+.0571	−.0113	.802	−.198	−4169.
20000	−.0596	−.0556	−.0040	.933	.067	−.0455	−.0314	−.0065	−.0249	.207	.793	−3878.
19	−.0566	−.0514	−.0052	.909	.091	−.0308	−.0049	+.0131	−.0181	....	......	−3587.
18	−.0536	−.0467	−.0069	.871	.129	−.0237	+.0063	+.0220	−.0157	.286	.714	−3256.
17	−.0506	−.0413	−.0093	.816	.184	−.0336	−.0165	+.0098	−.0263	....	......	−2879.
16	−.0476	−.0368	−.0108	.773	.227	−.0285	−.0094	+.0104	−.0197	....	......	−2567.
15000	−.0446	−.0317	−.0129	.710	.290	−.0611	−.0775	−.0448	−.0327	.578	+.422	−2210.
14	−.0416	−.0268	−.0148	.644	.356	−.0424	−.0431	−.0100	−.0331	.232	+.768	−1869.
13	−.0386	−.0235	−.0151	.608	.392	−.0719	−.1052	−.0738	−.0314	.701	+.299	−1637.
12	−.0356	−.0204	−.0152	.574	.426	−.1159	−.1962	−.1537	−.0425	.783	+.217	−1426.
11	−.0327	−.0180	−.0147	.551	.449	−.1744	−.3162	−.2494	−.0668	.789	+.211	−1255.
10000	−.0297	−.0164	−.0133	.554	.446	−.1729	−.3162	−.2626	−.0536	.831	+.169	−1145.
9	−.0267	−.0149	−.0118	.559	.441	−.2174	−.4081	−.3315	−.0766	.812	+.188	−1039.
8	−.0237	−.0139	−.0098	.586	.414	−.2442	−.4647	−.3775	−.0872	.813	+.187	− 969.
7	−.0207	−.0121	−.0086	.586	.414	−.2765	−.5323	−.4547	−.0776	.854	+.146	− 846.
6	−.0177	−.0110	−.0067	.620	.380	−.2728	−.5279	−.4201	−.1078	.796	+.204	− 764.
5000	−.0147	−.0103	−.0044	.703	.297	−.3252	−.6357	−.5437	−.0920	.855	+.145	− 721.
4000	−.0117	−.0092	−.0025	.787	.213	−.3439	−.6761	−.5411	−.1350	.800	+.200	− 643.
3000	−.0044	−.0036	−.0008	.826	.174	−.0543	−.1042	−.0594	−.0448	.570	+.430	− 251.
2500	−.0036	−.0023	−.0013	.640	.360	+.0110	+.0256	+.0283	−.0027	1.105	−.105	− 163.
2000	−.0029	−.0014	−.0015	.463	.537	−.1270	−.2511	−.1938	−.0573	.772	+.228	− 94.
1500	−.0022	−.0007	−.0015	.315	.685	−.1398	−.2774	−.2225	−.0549	.802	+.198	− 48.
1000	−.0015	−.0002	−.0013	.158	.842	−.2004	−.3993	−.3309	−.0684	.829	+.171	− 16.
500	.0000	.0000	.0000	....	....	−.0504	−.1008	−.0763	−.0245	.757	+.243	− 2.
Surface	....	....	....	....	....	....	....	....	....	....	......	....
Sums	−3.6942	−2.7841	−0.9101	....	....	−3.8181	−3.9419	−2.5744	−1.3675	....	......	....

−1.9120 −1.7177 −0.1943. The summations from the surface to 35500 give the amounts measured by the pyrheliometer.

(1) $J_0 = cT^4$ (2) $J_a = cT^a$. (3) $D = 2J_0 - E$. (4) $B = D - J_a$.

84, in $\frac{gr.\ cal.}{cm.^2\ min.}$ units. The values of $(Q_1 - Q_0)$ in M. K. S. mechanical units are found in the last column. The summations are as follows for the entire depth of the atmosphere.

Total sum of solar radiation,	$\Sigma (E_1 - E_0)$	$= - 3.6942$	$\frac{gr.\ cal.}{cm.^2 min.}$
Transmitted solar radiation,	$\Sigma (Q_1 - Q_0)$	$= - 2.7177$	"
Absorbed solar radiation,	$\Sigma (A_1 - A_0)$	$= - 0.9101$	"
Atmospheric (black) radiation,	$\Sigma (J_{0 \cdot 1} - J_{0 \cdot 0})$	$= - 3.8181$	"
Total terrestrial radiation,	$\Sigma (D_1 - D_0)$	$= - 3.9419$	"
Transmitted terrestrial radiation,	$\Sigma (B_1 - B_0)$	$= - 2.5744$	"
Absorbed terrestrial radiation	$\Sigma (J_{a \cdot 1} - J_{a \cdot 0})$	$= - 1.3675$	"

The minus (−) sign is due to the fact that while the positive (+) direction is along the axis z outward, the temperature and thermal gradients decrease in this direction; a positive (+) sign would indicate an inversion of temperatures and thermal quantities outward. In the case of p, k, the coefficients are all positive; in the case of p^1, k^1 they are generally positive in the lower levels, but sometimes (+) and sometimes (−) occur in the upper levels. Since the original observations of the velocity are lacking, those used being supplied by analogy, it is probable that the data of Huron, September 1, 1910, are inadequate to produce accurate values of p^1, k^1.

II. Second Distribution of Temperature

The Balloon Ascension, Uccle, November 9, 1911

As already stated the gravity residuals in the check equation

$$(196) \qquad g\,(z_1 - z_0) = - \frac{P_1 - P_0}{\rho_{10}} - (Cp_a - Cp_{10})\,(T_a - T_0) + \Delta\, g\,(z_1 - z_0)$$

were quite large above 40,000 meters in the Huron ascension. It was supposed that this was due to the fact that the velocities q were assumed, that g was taken constant in the adiabatic gradient, and that the formulas might begin to fail when the P, ρ terms became very small. Accordingly, a new series of computations has been undertaken for Uccle, June 9, September 13, November 9, 1911, the necessary data including velocities having been courteously supplied by Dr. Vincent, Directeur de l'Institut Royal Météorologique de Belgique. In these, besides using the observed velocities, the gravity ranges with the height. It is known that a temperature T introduced into the series of computations, building upward from the surface from level to level, whether observed or assumed, can always be checked by the control gravity equation. If the series of T is correct, $\Delta g (z_1 - z_0)$ vanishes. Hence, it is practical, by a set of trial computations, to arrive at such temperatures throughout the atmosphere as will satisfy the entire group of thermodynamic equations. The balloon ascensions to 20,000 or 30,000 meters give the necessary foundations upon which the entire structure can be built up to 90,000 meters. Up to 40,000 meters the four ascensions give about the same residuals, with occasional wide pairs depending on an erroneous T. The trials to reduce the check residuals were therefore limited to the region above 40,000 meters. It appears that positive residuals are to be diminished by increasing the temperatures on the several levels. The general outcome was to introduce a second temperature region beginning at about $T = 170°$ on the 40,000-meter plane and terminating near the 90,000-meter plane. The remarkable result is that the formulas are rigorous, however small the values of P, ρ, R, may become for the successive T, and that small residuals above 50,000 meters imply rather large changes in the temperatures T. Extending the temperatures from 40,000 meters to levels higher than 50,000 meters, the residuals have the following mean values:

Huron,	Sept.	1,	1910, 40,000 to 55,000,	three trials,	$\Delta g (z_1 - z_0)$	=	+163.0
Uccle,	June	9,	1911, 40,000 to 69,000,	four	" "	=	+ 17.8
Uccle,	Sept.	13,	1911, 40,000 to 80,000,	four	" "	=	+ 4.8
Uccle,	Nov.	9,	1911, 40,000 to 90,000,	two	" "	=	+ 2.5

Table 85 contains a summary of the Uccle, Nov. 9, 1911, ascension, where the gravity differences may be examined in the

TABLE 85

Extension of the Thermodynamic Data

From 40,000 Meters to 90,000 Meters. Balloon Ascension, Uccle, November 9, 1911

z	T	P	ρ	C_p	$\Delta g(z_1-z_0)$	W_1-W_0	U_1-U_0	K_{10}	$\log c$	a
Height, in Meters	Temperature	Pressure	Density	Specific Heat Ratio	Gravity Difference Check	External Work of Expansion	Inner Kinetic Energy	Radiation Energy	Coeffic. in the Law	Exponent in the Law
90000	0.0	1.8459×10^{-15}	8.5774×10^{-17}	.07461	+0.7	1.0	−9527.0	-8.171×10^{-13}	−12.550	0.500
89	4.0	1.8323×10^{-13}	3.0663×10^{-13}	0.5179	+1.8	4.8	−9527.8	-3.507×10^{-9}	−10.090	1.200
88	8.0	5.7840×10^{-11}	1.8360×10^{-11}	1.3653	+2.3	11.0	−9528.0	-1.925×10^{-7}	− 9.450	1.820
87	12.0	1.6778×10^{-9}	2.0088×10^{-10}	2.4078	+2.2	18.9	−9528.3	-2.842×10^{-6}	− 9.902	2.160
86	16.0	1.8304×10^{-8}	1.0991×10^{-9}	3.6018	+1.9	27.9	−9528.4	-1.430×10^{-5}	− 8.252	2.319
85000	20.0	1.1691×10^{-7}	4.1087×10^{-9}	4.9237	+2.1	37.7	−9527.7	-5.935×10^{-5}	− 8.512	2.436
84	24.0	5.3219×10^{-7}	1.2072×10^{-8}	6.3567	+2.0	48.2	−9526.9	-1.924×10^{-4}	− 8.698	2.522
83	28.0	1.9177×10^{-6}	3.0032×10^{-8}	7.8906	+1.6	59.3	−9526.1	-5.239×10^{-4}	− 8.881	2.605
82	32.0	5.8231×10^{-6}	6.6162×10^{-8}	9.5172	+1.7	71.0	−9524.6	-1.256×10^{-3}	− 7.020	2.668
81	36.0	1.5516×10^{-5}	1.3282×10^{-7}	11.228	+1.4	83.2	−9523.2	-2.726×10^{-3}	− 7.143	2.724
80000	40.0	3.7297×10^{-5}	2.4782×10^{-7}	13.020	+1.4	97.7	−9521.3	-5.424×10^{-3}	− 7.251	2.773
79	43.0	8.3252×10^{-5}	4.3864×10^{-7}	15.273	+1.5	112.0	−9516.0	-1.046×10^{-2}	− 7.365	2.825
78	46.0	1.7606×10^{-4}	7.4717×10^{-7}	17.726	+1.5	131.8	−9514.3	-1.708×10^{-2}	− 7.442	2.860
77	49.0	3.5522×10^{-4}	1.2308×10^{-6}	20.381	+1.7	151.0	−9509.8	-3.124×10^{-2}	− 7.543	2.906
76	52.0	6.8738×10^{-4}	1.9682×10^{-6}	23.241	+1.8	171.6	−9504.8	-5.224×10^{-2}	− 7.622	2.942
75000	55.0	1.2823×10^{-3}	3.0664×10^{-6}	26.310	+1.7	193.7	−9499.4	-8.568×10^{-2}	− 7.710	2.982
74	58.0	2.3148×10^{-3}	4.6672×10^{-6}	29.591	+1.9	217.0	−9493.2	− .13285	− 7.776	3.012
73	61.0	4.0561×10^{-3}	6.9643×10^{-6}	33.085	+1.9	242.1	−9486.6	− .20874	− 7.851	3.046
72	64.0	6.9198×10^{-3}	1.0169×10^{-5}	36.796	+1.9	268.8	−9479.2	− .31699	− 7.915	3.075
71	67.0	1.1521×10^{-2}	1.4611×10^{-5}	40.725	+1.7	297.2	−9471.6	− .47209	− 7.977	3.103
70000	70.0	.018766	2.0671×10^{-5}	44.879	+1.9	326.6	−9462.9	− .69198	− 6.036	3.130
69	73.0	.029943	2.8817×10^{-5}	49.254	+1.9	357.9	−9453.7	− .99724	− 6.093	3.156
68	76.0	.046898	3.9649×10^{-5}	53.856	+1.5	390.9	−9444.3	−1.4168	− 6.144	3.179

67	79.0	.072212	5.3898×10^{-5}	58.691	+2.8	424.9	−9433.1	− 1.9880	− 6.197	3.208
66	82.0	.109890	7.2410×10^{-5}	63.758	+1.8	461.5	−9422.5	− 2.7511	− 6.245	3.225
65000	85.0	.16333	9.6292×10^{-5}	69.053	+2.2	498.7	−9410.3	− 3.7680	− 6.291	3.246
64	88.0	.24047	1.2678×10^{-4}	74.587	+2.0	538.8	−9398.1	− 5.1007	− 6.333	3.265
63	91.0	.34957	1.6542×10^{-4}	80.360	+1.5	579.5	−9375.5	− 6.8307	− 6.377	3.285
62	94.0	.50213	2.1402×10^{-4}	86.376	+2.3	621.9	−9370.9	− 9.0866	− 6.417	3.303
61	97.0	.71307	2.7464×10^{-4}	92.650	+1.9	666.7	−9356.7	− 11.956	− 6.456	3.321
60000	100.0	1.0021	3.4983×10^{-4}	99.130	+1.5	717.7	−9341.1	− 15.546	− 6.498	3.340
59	102.0	1.3965	4.4296×10^{-4}	106.97	+3.0	773.5	−9321.7	− 19.987	− 6.540	3.359
58	104.0	1.9336	5.5829×10^{-4}	115.25	+6.9	829.4	−9297.6	− 25.522	− 6.582	3.378
57	106.0	2.6622	7.0083×10^{-4}	124.02	+3.8	895.4	−9289.2	− 32.594	− 6.621	3.396
56	108.0	3.6417	8.7570×10^{-4}	133.26	+2.5	963.0	−9257.5	− 41.225	− 6.661	3.414
55000	110.0	4.9551	1.0901×10^{-3}	143.00	+2.1	1033.6	−9233.6	− 51.979	− 6.698	3.431
54	112.0	6.7064	1.3519×10^{-3}	153.28	+4.5	1105.2	−9205.4	− 65.380	− 6.736	3.448
53	114.0	9.0252	1.6697×10^{-3}	164.08	+1.5	1185.9	−9181.1	− 81.660	− 6.773	3.465
52	116.0	12.091	2.0557×10^{-3}	175.47	+4.7	1264.5	−9148.9	− 101.82	− 6.808	3.481
51	118.0	16.111	2.5212×10^{-3}	187.41	+3.3	1351.7	−9119.7	− 126.33	− 6.843	3.497
50000	120.0	21.871	3.0822×10^{-3}	199.96	+5.4	1431.0	−9085.3	− 156.64	− 6.879	3.513
49	123.0	28.189	3.7531×10^{-3}	211.84	+4.5	1512.5	−9057.2	− 194.98	− 6.905	3.525
48	126.0	36.929	4.5477×10^{-3}	223.05	+3.4	1597.0	−9028.3	− 239.93	− 6.932	3.537
47	129.0	48.084	5.4867×10^{-3}	235.12	+4.7	1681.4	−8996.1	− 294.67	− 6.961	3.550
46	132.0	62.226	6.5907×10^{-3}	247.54	+3.8	1770.7	−8965.0	− 359.90	− 6.987	3.562
45000	135.0	80.076	7.8853×10^{-3}	260.34	+4.6	1850.7	−8931.0	− 439.09	− 5.012	3.573
44	139.0	102.39	9.3916×10^{-3}	271.45	+2.9	1920.6	−8904.3	− 536.73	− 5.030	3.581
43	144.0	129.93	1.1125×10^{-2}	280.69	+2.4	1976.0	−8881.7	− 656.63	− 5.043	3.587
42	150.0	163.42	1.3095×10^{-2}	287.91	+2.6	2007.0	−8863.9	− 804.77	− 5.052	3.591
41	158.0	203.42	1.5302×10^{-2}	291.19	+4.4	1991.8	−8845.8	− 991.80	− 5.050	3.590
40000	169.0	250.04	1.7721×10^{-2}	288.95	+4.4	1919.6	−8864.9	−1251.4	− 5.030	3.581

The check equation is $g\,(z_1 - z_0) = -\dfrac{P_1 - P_0}{\rho_{10}} - (Cp_a - Cp_{10})\,(T_a - T_0) + \Delta\, g\,(z_1 - z_0)$

sixth column. They are smaller from 90,000 to 65,000 than from 65,000 to 40,000 meters. The inference is that the entire thermodynamic system up to vanishing quantities is reliable for numerous researches depending upon the data. In order to illustrate how small these quantities really are, we have at 65,000 meters, $B = 0.001225$ millimeter of mercury, $\rho = 0.00000009629$ C.G.S., while hydrogen at normal surface conditions is $\rho_H = 0.00008924$ $gr./cm^3$. The remarkable precision of the computations is proved by the system of checks from the surface to 90,000 meters.

These data above 40,000 meters, therefore, modify the data of Table 82, and may be substituted for them. The results of further studies on these computations will be published in Bulletin No. 4 of the Argentine Meteorological Office. It may be here remarked that the thermal efficiency of the atmosphere begins at about 65,000, there being little absorption above it. The shape of the absorbing area above 40,000 meters has been changed by this extension, but the amount is not very different, as will be illustrated. Compare Figs. 74, 75, 79.

Summary of the Computations for Twenty-one Balloon Ascensions

In order to improve the data as far as practical, similar computations were extended to twenty other balloon ascensions.

(10) United States, Omaha, February 21, 22, 23, 1911; September 28, 1909. Huron, September 1, 4, 7, 16, 1910. Indianapolis, October 6, 30, 1909.

(6) Europe, Lindenburg, July 27, 1908; April 27, May 5, 6, September 2, 1909. Milan, September 7, 1907.

(5) Atlantic Tropics, *Otaria*, June 19, 1906; July 29, August 29, September 9, 25, 1907. The mean values for the several quantities appear on Table 86, where the data can be conveniently examined.

$(E_1 - E_0)$. The same values were adopted throughout, but since balloon ascensions now reach 28,000 meters the critical value at that elevation can be further examined.

$(Q_1 - Q_0)$, $(A_1 - A_0)$. These areas are seen on Fig. 75, and they have been described. It should be noted that the course of $(Q_1 - Q_0)$ in the lower absorption region does not in the least follow the distribution of the aqueous vapor contents, which is at a minimum where $(Q_1 \dot{-} Q_0)$ is at its maximum. This problem is very difficult to solve satisfactorily.

p increases rapidly up to 2,000 meters, then more slowly up to $p = 1.000$ in the true isothermal level at 27,000 meters; above that level p falls to a small value $p = 0.053$ at 46,000 meters, and then rises to about unity on the vanishing plane.

k passes through inverse relations in respect of p.

J_0 has a maximum value at 4,000 meters, and proceeds irregularly to zero at the top. Its value can always be recovered from $J_0 = \frac{1}{2}(D + E)$ in the tables.

D has a large negative maximum at the surface, passes through zero with change of sign near 18,000 meters, and gradually increases to its maximum $+ 0.1406$ on the vanishing plane.

B falls from a negative maximum at the surface to zero at the 13,000-meter level, at the bottom of the so-called isothermal layer, and then increases irregularly to the top of the atmosphere.

J_a begins with a maximum at the surface and gradually falls to a small quantity, finally vanishing at the top.

p^1 has a nearly constant value of 0.800 in the convectional region and probably about 0.500 above it. k^1 is correlative to p^1.

Uccle, November 9, 1911

The data are fully computed for the Uccle ascensions, as extended to 90,000 meters, and the results above 40,000 meters will be briefly summarized. The total atmospheric radiation energy,

$$\Sigma J_0 = \Sigma (c_1 T_1^4 - c_0 T_0^4)$$

from the surface to 90,000 meters, is equivalent to the "solar constant" at the distance of the earth, because it represents the amount of heat required to maintain the existing temperature distribution in equilibrium with the incoming and outgoing radiations that are in operation day and night. Summarizing the J_0 data, we have,

TABLE 86

SUMMARY FOR 21 BALLOON ASCENSIONS. THE TOTAL, TRANSMITTED AND ABSORBED QUANTITIES AND THE COEFFICIENTS

Solar radiation in $\frac{gr.\ cal.}{cm^2.\ min.}$

z Height, in meters	E_1-E_0	Q_1-Q_0			A_1-A_0			p			k		
	General Value for These Stations	U. S. 10	Europe 6	A. T. 5	U. S. 10	Europe 6	A. T. 5	U. S. 10	E'rope 6	A. T. 5	U. S. 10	E'rope 6	A T. 5
26000	−.0775	−.0780	......	......	+.0005	......	......	1.004	....	....	−.004	....	....
25000	−.0745	−.0747	......	......	+.0002	......	......	1.002	....	....	−.002	....	....
24	−.0715	−.0714	......	......	−.0001	......	......	.997	....	....	.003	....	....
23	−.0686	−.0679	......	......	−.0007	......	......	.991	....	....	.009	....	....
22	−.0656	−.0643	......	......	−.0013	......	......	.980	....	....	.020	....	....
21	−.0626	−.0603	......	......	−.0023	......	......	.963	....	....	.037	...	....
20000	−.0596	−.0568	......	......	−.0028	......	......	.950	....	....	.050	....	....
19	−.0566	−.0527	......	......	−.0039	......	......	.933	....	....	.067	....	....
18	−.0536	−.0483	......	......	−.0053	......	......	.901	....	....	.099	....	....
17	−.0506	−.0425	......	......	−.0081	......	......	.841	....	....	.159	....	....
16	−.0476	−.0387	−.0399	−.0367	−.0089	−.0077	−.0109	.813	.841	.774	.187	.159	.226
15000	−.0446	−.0336	−.0342	−.0271	−.0110	−.0104	−.0175	.754	.769	.610	.246	.231	.390
14	−.0416	−.0291	−.0279	−.0232	−.0125	−.0137	−.0184	.702	.673	.558	.298	.327	.442
13	−.0386	−.0260	−.0244	−.0213	−.0126	−.0142	−.0173	.672	.632	.530	.328	.368	.470
12	−.0356	−.0232	−.0204	−.0198	−.0124	−.0152	−.0158	.652	.555	.556	.348	.445	.444
11	−.0327	−.0190	−.0170	−.0186	−.0137	−.0157	−.0141	.582	.548	.572	.418	.452	.428
10000	−.0297	−.0155	−.0165	−.0179	−.0142	−.0132	−.0118	.522	.559	.604	.478	.441	.396
9	−.0267	−.0130	−.0162	−.0171	−.0137	−.0105	−.0096	.487	.607	.635	.513	.393	.365
8	−.0237	−.0119	−.0148	−.0157	−.0118	−.0089	−.0080	.497	.593	.666	.503	.407	.334
7	−.0207	−.0104	−.0132	−.0143	−.0103	−.0075	−.0064	.500	.637	.667	.500	.363	.333
6	−.0177	−.0090	−.0113	−.0125	−.0087	−.0064	−.0052	.508	.639	.708	.492	.361	.292
5000	−.0147	−.0074	−.0098	−.0100	−.0073	−.0049	−.0047	.502	.665	.683	.498	.335	.317
4000	−.0117	−.0060	−.0076	−.0077	−.0057	−.0041	−.0040	.508	.648	.632	.492	.352	.368
3000	−.0044	−.0024	−.0029	−.0029	−.0020	−.0015	−.0015	.546	.605	.711	.454	.395	.289
2500	−.0037	−.0018	−.0025	−.0025	−.0019	−.0012	−.0012	.502	.616	.690	.498	.384	.310
2000	−.0030	−.0012	−.0020	−.0018	−.0018	−.0010	−.0012	.398	.594	.581	.602	.406	.419
1500	−.0022	−.0005	−.0014	−.0009	−.0017	−.0008	−.0013	.272	.548	.386	.728	.452	.614
1000	−.0015	−.0001	−.0006	−.0002	−.0014	−.0009	−.0013	.118	.473	.233	.882	.527	.767
500	−.0007	.0000	−.0003	−.0001	−.0007	−.0004	−.0006	.080	.301	.171	.920	.699	.829
Surface	.0000	.0000	.0000	.0000	.0000	.0000	.0000	....	....	....	....	....	....

Terrestrial radiation in $\frac{gr.\ cal.}{cm^2.\ min.}$

s	D_1-D_0			B_1-B_0			$Ja._1-Ja._0$			p^1			k^1		
	U. S.	Europe	A. T.	U. S.	Europe	A. T.	U. S.	Europe	A. T.	U. S.	E'rope	A. T.	U. S.	E'rope	A. T.
26000	+.0268	....	....	+.0313	....	....	−.0045	....	....	.856	....	....	−.144	....	....
25000	+.0327	....	....	+.0439	....	....	−.0112	....	....	.745	....	....	−.255	....	....
24	+.0231	....	....	+.0348	....	....	−.0117	....	....	.663	....	....	−.337	....	....
23	+.0090	....	....	+.0334	....	....	−.0244	....	....	.269	....	....	−.731	....	....
22	+.0278	....	....	+.0410	....	....	−.0132	....	....	.678	....	....	−.322	....	....
21	+.0179	....	....	+.0331	....	....	−.0152	....	....	.540	....	....	−.460	....	....
20000	+.0042	....	....	+.0229	....	....	−.0187	....	....	.184	....	....	−.816	....	....
19	+.0001	....	....	+.0241	....	....	−.0240	....	....	.004	....	....	−.996	....	...
18	+.0079	....	....	+.0224	....	....	−.0145	....	....	.353	....	....	−.647	....	....
17	−.0157	....	....	+.0043	....	....	−.0200	....	....	−.274	....	....	−.726	....	....
16	−.0370	−.0159	−.0639	−.0126	+.0087	−.0475	−.0244	−.0246	−.0164	.607	.793	.606	.393	.207	.394
15000	−.0665	−.0419	−.1250	−.0473	−.0142	−.0940	−.0192	−.0277	−.0310	.662	.479	.733	.339	.521	.267
14	−.1271	−.0223	−.1075	−.0860	+.0022	−.0709	−.0411	−.0245	−.0366	.607	.628	.715	.393	.372	.285
13	−.1086	−.0876	−.1585	−.0723	−.0605	−.1287	−.0363	−.0271	−.0298	.565	.704	.808	.435	.296	.192
12	−.1561	−.1433	−.3886	−.1107	−.1078	−.3223	−.0454	−.0355	−.0663	.684	.694	.792	.316	.306	.208
11	−.1898	−.3351	−.4062	−.1372	−.2587	−.3326	−.0526	−.0764	−.0736	.730	.744	.819	.270	.256	.181
10000	−.2590	−.4029	−.4765	−.1989	−.3324	−.3906	−.0601	−.0705	−.0859	.734	.806	.820	.266	.194	.180
9	−.3433	−.4368	−.4436	−.2731	−.3584	−.3623	−.0702	−.0784	−.0813	.784	.815	.816	.216	.185	.184
8	−.4180	−.4620	−.4957	−.3293	−.3748	−.4043	−.0887	−.0872	−.0914	.815	.810	.815	.185	.190	.185
7	−.5484	−.4761	−.4744	−.4440	−.3846	−.3864	−.1044	−.0915	−.0880	.813	.808	.815	.187	.192	.185
6	−.4643	−.4944	−.4946	−.2893	−.3984	−.4008	−.1750	−.0960	−.0938	.790	.785	.809	.210	.215	.191
5000	−.5352	−.4404	−.4060	−.3346	−.3648	−.3248	−.2006	−.0756	−.0812	.820	.792	.800	.180	.208	.200
4000	−.6171	−.5371	−.5165	−.3848	−.4325	−.4298	−.2323	−.1046	−.0867	.797	.804	.806	.203	.196	.194
3000	−.2733	−.2286	−.2969	−.1285	−.1805	−.2406	−.1448	−.0481	−.0563	.757	.791	.811	.243	.209	.189
2500	−.2090	−.3261	−.2779	−.1074	−.2572	−.2284	−.1016	−.0689	−.0495	.728	.776	.827	.272	.224	.173
2000	−.2216	−.2769	−.1919	−.1387	−.2155	−.1538	−.0829	−.0614	−.0381	.687	.781	.797	.313	.219	.203
1500	−.2336	−.2944	−.3067	−.1075	−.2336	−.2453	−.1261	−.0608	−.0614	.757	.783	.795	.243	.217	.205
1000	−.4400	−.1404	−.2579	−.2410	−.1146	−.2077	−.1990	−.0258	−.0502	.806	.810	.815	.194	.190	.185
500	−.2129	−.3696	−.5041	−.1239	−.3035	−.4486	−.0890	−.0661	−.0555	.798	.822	.847	.202	.178	.153
Surface	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....

From	the	76,000	to	90,000	meters, ΣJ_0 =		−0.0001
"	"	66,000	"	76,000	"		−0.0032
"	"	56,000	"	66,000	"		−0.0372
"	"	46,000	"	56,000	"		−0.1981
"	"	36,000	"	46,000	"		−0.2952
"	"	26,000	"	36,000	"		−0.2914
"	"	16,000	"	26,000	"		−0.3275
"	"	6,000	"	16,000	"		−1.0151
"	"	Surface	"	6,000			−1.5658
		Total atmospheric radiation energy					−3.7336

From the Huron ascension we obtained −3.8181. It is apparently a question of distribution rather than the amount that is concerned in these two temperature systems. The total "solar constant" derived from the black atmospheric radiation is about *twice the amount measured by the pyrheliometer.* More data on this important subject will be accumulated.

First Method of Computing the Pyrheliometric Data. The Effective Energy of Radiation and the Solar Constant.

When the preceding Chapter V, on radiation, was written it was supposed that the extension of these computations would be limited to the lower levels in the atmosphere where direct temperatures were obtained in balloon ascensions. But it is now evident that temperatures may be measured, or assumed by trial, and they will be correct provided they produce such values of $P.\ \rho.\ R.\ T.$ in successive stages as will satisfy the gravity equation with small residuals,

$$(196) \quad g\,(z_1 - z_0) = -\frac{P_1 - P_0}{\rho_{10}} - \tfrac{1}{2}\,(q_1^2 - q_0^2) - (Q_1 - Q_0).$$

Hence, by a series of trial computations, it has become possible to extend the data up to 90,000 meters, and four balloon ascensions have been computed up to 70,000 or 90,000 meters. We thus obtain by (335), (333)

$$(Q_1 - Q_0) = (W_1 - W_0) + (U_1 - U_0) = P_{10}\,(v_1 - v_0) + (U_1 - U_0),$$

$$(339) \quad \frac{Q_1 - Q_0}{v_1 - v_0} - P_{10} = \frac{U_1 - U_0}{v_1 - v_0} = K_{10} = c_1\,T_1^{a_1} - C_0\,T_0^{a_0} = J_a$$

for *selective* radiation, and

(344) $$J_0 = c_1 T_1^4 - c_0 T_0^4$$

for *black-body* radiation, in each 1,000-meter stratum from the surface upward, and thence the total thermodynamic and radiation energies throughout the atmosphere by taking the summations for all of the strata. The results are summarized in Tables 82, 85, and Figs. 75, 79. Similarly, the computations include the data of the Kinetic Theory of Gases, (H, U) external and internal energies; $(q. \gamma)$ velocities; $(n. N)$ number of molecules, l_{max} free path length, ν number of collisions, m_H mass of the hydrogen atom, e_- the negative ion charge, in all strata to 90,000 meters. Tables 96, 97. See Bulletin No. 4, O. M. A.

TABLE 87

SUMMARY OF THE RESULTS OF THE COMPUTATIONS ON THREE BALLOON ASCENSIONS FOR THE VALUES OF THE THERMODYNAMIC AND RADIATION ENERGIES

Station and Date of the Observations Height in Meters of 0° Temperature	Uccle June 9, 1911 70,000	Uccle Sept. 13, 1911 80,000	Uccle Nov. 9, 1911 90,000
$\Sigma[g(z_1-z_0)]$ gravity acceleration....	9.5792	11.0895	12.4565
$\Sigma[-(\frac{1}{2}q_1^2-q_0^2)]$ kinetic energy of circulation...........	0.0000	−0.0006	0.0002
$\Sigma[-(Q_1-Q_0)]$ free heat (non-adiabatic)...............	5.4250	6.6676	8.6590
$\Sigma[g(z_1-z_0)+\frac{1}{2}(q_1^2-q_0^2)+(Q_1-Q_0)]$ summary....	4.1542	4.4219	3.7975
$\Sigma\left[-\frac{P_1-P_0}{\rho_{10}}\right]$ hydrostatic pressure.	4.1146	4.1608	3.9386
$\Sigma[J_0 = c_1T_1^4 - c_0T_0^4]$ black body radiation.............	3.9690	3.9261	3.5373
$\Sigma[K_1-K_0]$ radiation energy........	1.4528	1.4672	1.4802
$\Sigma\left[\frac{Q_1-Q_0}{v_1-v_0}\right]$ free heat per volume change..............	0.0000	−0.0087	0.0352
$\Sigma[P_1-P_0]$ pressure differences......	1.4602	1.4404	1.4085
$\Sigma[J_a = c_1T_1^{a1} - c_0T_0^{a0}]$ selective radiation.............	1.4603	1.4600	1.4395

General mean of the thermodynamic data 4.0979
General mean of the black body radiation data J_0 . . . 3.9476
General mean of the radiation and pressure data 1.4516
General mean of the selective radiation data J_a 1.4536

Table 87 shows that the thermodynamic state of the earth's atmosphere is such that about 4.00 gr. cal./cm.[2] min. is required to raise it from the frozen and solid state of air at 0° *A* into the expanded state it now has with its *P. ρ. R. T.* in all strata. This is, also, shown to be equal to the black body radiation J_0 as derived from the thermodynamic data. The selective absorption J_a in a state for continuous emission is about 1.46 calories, and this is equal to the temporary amount of energy in the $(K_1 - K_0)$ and $(P_1 - P_0)$ terms.

Table 88 contains a summary of the terms in the equation of equilibrium, (785) $S = I_1 + B + J_a$ *and* $I_1 = R$.

TABLE 88

THE TERMS IN THE "EFFECTIVE" RADIATION FORMULA, $I_1 = S - R - J_a$, WHERE $I_1 = R$

z	S	J_a	$I_1 = R$	p	p_w	m_e
90000	4.00	0.000	2.000	1.000	...	14000
76000	4.00	0.000	2.000	1.000	...	10000
66000	4.00	0.000	2.000	1.000	...	10000
56000	4.00	0.000	1.999	0.999	...	10000
46000	4.00	0.007	1.993	0.998	...	10000
36000	4.00	0.034	1.983	0.996	...	6000
36000	4.00	0.070	1.967	0.994	...	3000
27000	4.00	0.100	1.950	0.992	...	3000
24000	4.00	0.130	1.935	0.991	...	3000
21000	4.00	0.170	1.915	0.988	...	3000
18000	4.00	0.220	1.890	0.984	...	3000
15000	4.00	0.295	1.853	0.975	...	3000
12000	4.00	0.410	1.795	0.961	...	3000
9000	4.00	0.580	1.710	0.950	...	3000
6000	4.00	0.776	1.617	0.929	0.920	3000
3000	4.00	1.050	1.475	0.887	0.890	3000
000	4.00	1.461	1.270	0.840	0.840	...

I_1 = the "effective" radiation = 2.00 at 90,000 meters.

S = the solar constant = 4.00 calories.

R = the "reflected" radiation = 2.00 at 90,000 meters, and neutralizes one-half the solar constant. $I_1 = R$ on all levels, except during changes in T.

J_a = the absorbed radiation in the lower levels, small at 40,000 meters, and increasing by variable values of p

in the Bouguer Formula, for m_0 about 3,000 meters, $I = I_0 \, p^{m}{}_{o}$.

p = the values of the coefficient of transmission formed by $p = \frac{I}{I_0}$ using the values of I for different 3,000-meters.

p_w = the general observed values of p at the sea level, at 3,000 meters as La Quiaca, with its probable value at 6,000 meters.

m_o = the adopted depth of the stratum in the zenith.

It is evident that p is not constant throughout the atmosphere, and that m_0 is not to be taken as unity for the 90,000 meters. On the other hand,

$$I = I_0 \, p_1^{m_1} \cdot p_2^{m_2} \cdot p_3^{m_3} \quad . \quad . \quad . \quad p_n^{m_n},$$

for variable (p. m.).

I_1 = the effective radiation as observed by the pyrheliometer. I_0 = 2.00 cal. at 90,000 meters.

I_0 penetrates to 30,000 meters undepleted, and then diminishes by a variable p, constant through m = 3,000 meters, in the Bouguer Formula, to 1.46 calories at the surface.

$R = I_1$ at every level. This return current neutralizes one-half of the solar constant S = 4.00 calories.

J_a = the absorbed energy in the lower levels, and together with R neutralizes $\frac{1}{2}$ S.

The Bolometer measures the transmitted parts of S.

The Pyrheliometer measures the transmitted parts of I_0, that is, one-half the "solar constant."

Figure 76 shows the relations of these terms throughout the atmosphere. The effective radiation I_0 proceeds downward till it is deflected by J_a acting outward, arriving at the surface as I_1. Simultaneously, the returning radiation $R = I_1$ in every level, during temperature equilibrium, reaches the top of the atmosphere, with the addition of J_a in the lower levels, having the value R = 2.00 cal. which neutralizes one-half of the incoming S = 4.00. It is easily seen that the pyrheli-

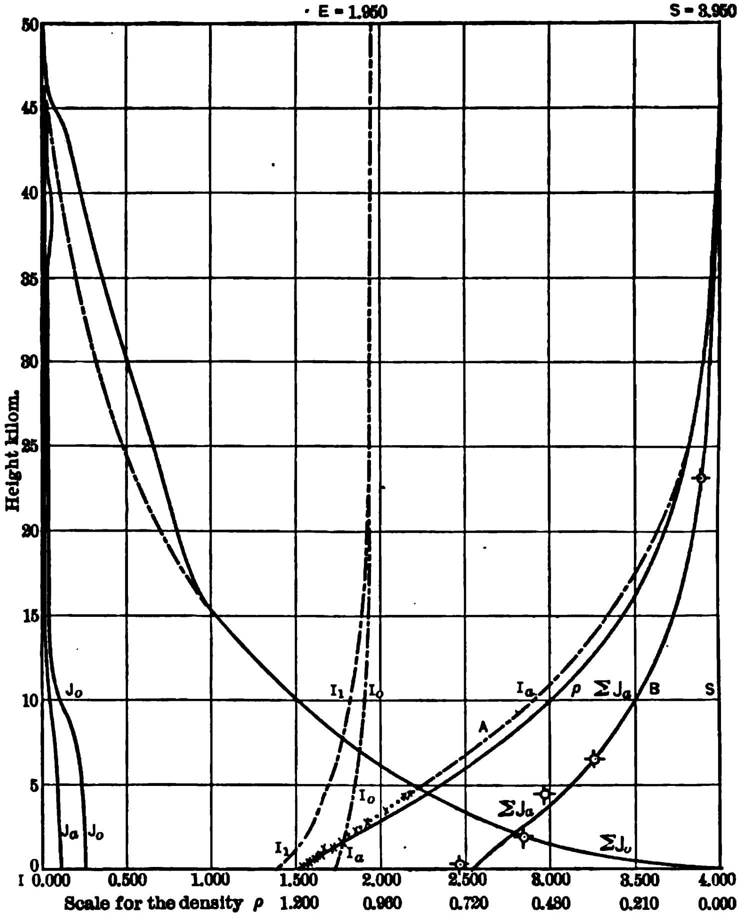


FIG. 76. Two methods of discussing the pyrheliometer observations.

For second method.

- S = the solar constant 3.95 gr. cal./cm.^{2} min.
- ΣJ_a = the radiation energy absorbed in producing temperature T.
- ρ = the density curve.
- A = the probable pyrheliometer curve.
- B = the probable bolometer curve.
- Area $S - B$ = the temperature radiation.
- Area B — A = the scattered radiation.
- Area $A - O$ = the free-heat radiation.
- J_a = absorbed energy in 1000 meters.
- J_o = black body energy in 1000 meters.
- I_1 = intensity of zenith sun.
- I_o = I_1 / p_w.
- I_a = I_1 / p_a.

ometer measures only the effective radiation $I_0 = \frac{1}{2} S$, while the bolometer measures such ordinates as are necessary for the construction of a 4.00 calories curve at solar temperature between 6,900° and 7,000°.

If we supply by simple interpolation the missing ordinates on the energy spectrum curve for 6,900°, so that the wave-length interval is $\Delta \lambda = 0.05 \mu$, and similarly fill out Abbot's ordinates for Washington, D. C. (34 *m*), Mt. Wilson (1,780 *m*), Mt. Whitney (4,420 *m*), (Bul. No. 3, O. M. A. Tables XXVI, XXVII), we have the following results for the sum of the observed ordinates, that is, the relative areas of transmission.

TABLE 89

COMPARISON OF THE ABSORBED ENERGY BY THE BOLOMETER AND THERMODYNAMIC DATA

Energy Spectrum	ΣJ	D	k Ratio	J_a	k^1 Ratio
6900°.	80.295	. .	. . .	4.00	. . .
Washington. . . . . . (34)	51.19	29.105	0.363	1.45	0.363
Mt. Wilson. . . . . . (1780)	59.55	20.745	0.258	1.15	0.287
Mt. Whitney. . . . . (4420)	61.38	18.915	0.236	0.90	0.225
Final curve. . . . . . (8000)	67.26	13.035	0.162	0.65	0.163

ΣJ is the sum of the ordinates of transmission at each station.
D is the absorbed part = ΣJ (6,900°) − ΣJ (station).
k is the ratio of absorbed part at each station to ΣJ (6,900°).
J_a is the computed thermodynamic absorption at same levels.
k^1 is the ratio of the absorbed part at each station to 4.00.

It is seen that these two ratios (k, k^1) are substantially the same, except for minor variations.

It must be concluded that the amount of absorption in the lower atmosphere J_a is about the same as the amount of depletion in the bolometric energy spectrum, referred to a 6,900° curve, or 4.00 calories for the solar constant. This analysis points to a very different method of discussing pyrheliometer observations from that commonly practised, and it promises to harmonize the several branches of this hitherto conflicting subject.

SUMMARY

1. The bolometer curves as observed are best satisfied by an energy spectrum of 6,900° or a solar constant of 4.00 gr. cal./cm.² min.

2. The thermodynamics of the atmosphere requires an expenditure of 4.00 calories to produce and maintain the existing pressures, densities, temperatures, and thermodynamic values up to 90,000 *m*.

3. The pyrheliometer measures only one-half the solar constant, that is, the efficient energy 2.00 calories, because one-half of the incoming radiation is neutralized by the returning energy stream of 2.00 calories.

4. The amounts of absorbed radiations, as measured by the bolometer ordinates or by the thermodynamic conditions, agree in giving about the same ratios (k, k^1).

5. The conclusions derived from the extensive computations summarized in this section verify, generally, the text in Chapter V.

Change of Theory

The foregoing analysis is based upon the view that the pyrheliometer measures the efficient incoming solar radiation at 2.00 calories, and upon the formula that half the incoming ray advances to the surface while half of it is scattered back to space. Further experience brings both these ideas into doubt, and we proceed to give evidence that the I_0 — curve of Fig. 76 should be made the I_a — curve, thus enclosing the scattering of radiation between the curves A and B.

The Second Method of Discussing Pyrheliometric Data

The discussion of pyrheliometer observations begun in Chapter V has been continued by the development of a new method of computation, which will be briefly described. Compute from the observed pyrheliometer readings the values of the intensity I_1, for the sun in the zenith, and with p_w, the uncorrected coefficient of transmission, extrapolate I_o the intensity reduced to the mean solar distance. Abbot multiplies I_o by a bolometer factor, 1.123 for Washington, and 1.094 for Mt. Wilson on the average, to which a small correction is then added for the effect of the aqueous vapor pressure e. We proceed to develop the data in another way. Collect the individual observations in convenient groups, according to the observed values of p_w, as (0.900 – 0.880), (0.880 – 0.860), . . . for I_o, I_1, e, and take the mean values, such as appear in Table 90 for Washington, D. C., at 34 meters, Bassour at 1,160 meters, La Quiaca at 3,465 meters. Plot on diagrams with p_w for abscissas, and values of I_0, I_1, in gr. cal./cm.^{2} min. for ordinates. It is seen that in the case of Washington, Fig. 77, that I_o is a line sloping downward to meet I_1 sloping upward in the contact point on the ordinate $I_a = 1.528$; in the case of Bassour I_o is a horizontal line meeting the upward sloping I_1 in the contact point 1.680; in the case of La Quiaca I_o is an upward sloping line to meet I_1 in the contact point 2.010. In each case the contact point is on the ordinate axis to which corresponds the coefficient of transmission $p_w = 1.000$, *which is that for perfect transmission.* Whatever may be the physical cause of the sloping of the I_o lines, below or above the horizontal, the contact point has eliminated that cause from the system. In Case III, Washington, it is necessary to depress the sloping line I_o into a horizontal position I_a; in Case II, Bassour, this is already done; in Case I, La Quiaca, the sloping line must be raised to the horizontal position I_a. In Case III, if $I_1/p_w = I_o$, then $I_1/p_a = I_a$, where p_a is *larger* than p_w; in Case I, if $I_1/p_w = I_o$, then $I_1/p_a = I_a$, where p_a is *smaller* than p_w. It is necessary to determine an equation for each station depending upon the aqueous vapor pressure e which will convert p_w into the required p_a. This is done in Section 2 of Table 90. Assume $p^1{}_w$ at convenient intervals 1.000, 0.980, 0.960, . . . ; take the contact point $I^1{}_o$ *from the diagram,* and $I^1{}_1$ from the mean line on the

TABLE 90

Examples of the Correction Equation in Computing the Pyrheliometric Observations

I	Washington, D. C. No.	p_w	I_0	I_1	e		Bassour No.	p_w	I_0	I_1	e	La Quiaca No.	p_w	I_0	I_1	e
...	7	.884	1.595	1.410	3.78	....	4	.908	1.627	1.477	6.77	18	.922	1.907	1.758	1.2
...	16	.874	1.604	1.402	....	....	13	.891	1.680	1.497	7.18	29	.915	1.893	1.732	1.2
...	20	.865	1.629	1.409	2.09	....	9	.872	1.688	1.472	7.59	27	.905	1.879	1.700	3.2
...	16	.854	1.630	1.392	....	....	8	.849	1.711	1.451	8.34	15	.895	1.862	1.666	2.5
...	22	.846	1.638	1.386	3.34	....	6	.827	1.672	1.383	8.53	7	.885	1.821	1.612	5.8
...	24	.834	1.691	1.410	4.06	....	10	.810	1.673	1.356	6.48	1	.869	1.803	1.549	6.2
...	17	.824	1.642	1.353	....	....	4	.788	1.663	1.311	8.43	1	.855	1.791	1.535	5.6
...	17	.815	1.740	1.418	3.69	....	7	.769	1.698	1.306	7.86	..	....	.....	.....	...
...	10	.804	1.659	1.349	5.02	....	5	.752	1.683	1.266	7.61	..	....	.....	.....	...
...	18	.784	1.679	1.316	....	....	3	.734	1.645	1.208	7.65	..	....	.....	.....	...
...	8	.773	1.683	1.303	....	....	2	.716	1.686	1.206	6.34	..	....	.....	.....	...
...	9	.764	1.747	1.335	5.19	....	..	....	.....	.....	....	..	....	.....	.....	...
...	11	.756	1.683	1.272	....	....	..	....	.....	.....	....	..	....	.....	.....	...
...	7	.743	1.693	1.260	7.03	....	..	....	.....	.....	....	..	....	.....	.....	...
...	5	.736	1.715	1.262	7.69	....	..	....	.....	.....	....	..	....	.....	.....	...

II	Washington, D. C. p^1_w	I^1_0	I^1_1	p_a	$e^1+2.4$	F	Bassour p^1_w	I^1_0	I^1_1	p_a	La Quiaca p^1_w	I^1_0	I^1_1	p_a	$e^1+5.0$	F
...	1.000	1.528	1.528	1.000	0.0	.....	1.000	1.680	1.680	1.000	1.000	2.010	2.010	1.000	0.0	....
...	.980	1.528	1.508	.986	0.8	.....	.980	1.680	1.648	.981	.980	2.010	1.944	.967	1.7	—.0079
...	.960	1.528	1.489	.973	1.5	.....	.960	1.680	1.614	.961	.960	2.010	1.878	.934	3.3	—.0080
...	.940	1.528	1.470	.961	2.3	.....	.940	1.680	1.580	.941	.940	2.010	1.812	.902	4.8	—.0080
...	.920	1.528	1.452	.949	3.0	.0097	.920	1.680	1.544	.919	.920	2.010	1.746	.869	6.4	—.0080
...	.900	1.528	1.434	.937	3.7	.0100	.900	1.680	1.529	.898	.900	2.010	1.680	.836	8.0	—.0080
...	.880	1.528	1.414	.924	4.5	.0098	.880	1.680	1.475	.878	.880	2.010	1.614	.803	9.6	—.0080
...	.860	1.528	1.394	.911	5.2	.0098	.860	1.680	1.441	.858	.860	2.010	1.548	.770	11.2	—.0080
...	.840	1.528	1.375	.899	6.0	.0098	.840	1.680	1.406	.837	.....	.....	.....	.....	....	....
...	.820	1.528	1.356	.886	6.7	.0099	.....	.....	.....	.....	.....	.....	.....	.....	....	....
...	.800	1.528	1.338	.875	7.4	.0101	.....	.....	.....	.....	.....	.....	.....	.....	....	....

Washington, D. C.: $p_a = p_w + 0.0100\,(e + 2.4)$; $F = \dfrac{p_a - p^1_w}{e^1 + 2.4}$

Bassour: $p_a = p_w$

La Quiaca: $p_a = p_w - 0.0080\,(e + 5.0)$

diagram; then compute $p_a = \frac{I^1_1}{I_o}$. Plot down e on an auxiliary ordinate scale for the same axis of abscissas, and draw the mean e. Now it is evident *that if the aqueous vapor effect is to be eliminated the line e should pass through the origin where* $p_w = 1.000$. In order to do this, draw a line through the origin parallel to e, that is $(e + 2.4)$ mm. for Washington, $(e - 7.5)$ mm. for Bassour, and $(e + 5.0)$ mm. for La Quiaca. Take $p_a - p_w = F\ (e + B)$, where $F =$ the coefficient, in fact the ratio of the ordinate of e to the ordinate of I_1 counted from the horizontal line I_a. Hence, we find the equation,

$$p_a = p_w + F\ (e + B),$$

where B is the vertical ordinate between the two vapor-pressure lines. In type II, $p_a = p_w$; in type III, $p_a = p_w + F\ (e + B)$; in type I, $p_a = p_w - F\ (e + B)$. At each station there is to be computed such a value of the coefficient of transmission p_a as will make the values of the extrapolated I_1 fall upon the horizontal line I_a, instead of upon the sloping line I_o. If they do fall upon a sloping line, it follows that the resulting mean values of a series, winter series, for example, will differ radically from the summer series, because the means will pertain to different groups of p_w. Take the mean values of each half of the groups in Section 1:

Station	p_w	I_0	p_w	I_0	Difference
Washington.....	0.854	1.633	0.772	1.687	+0.054 Type III
Bassour.........	0.869	1.676	0.763	1.674	−0.002 Type II
La Quiaca.......	0.914	1.893	0.876	1.819	−0.074 Type I

This is evidently one cause for the incessant variations that characterize pyrheliometer mean values. It is plain that many such fluctuations as have been attributed to solar action are, in fact, due to the imperfect elimination of the terrestrial effects of aqueous vapor and dust as well as density from the intensity of radiation at the station.

Summary of the Correction Equations

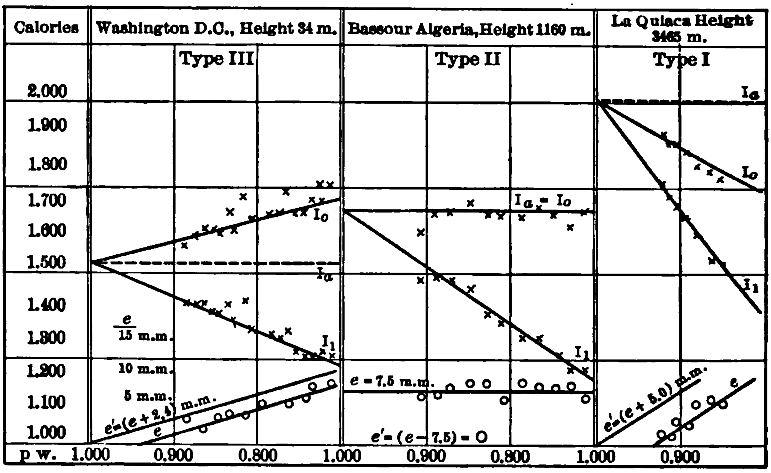


FIG. 77.—Three types of pyrheliometric data

Type I.

La Quiaca, height 3,465 meters, $p_a = p_w - 0.0080\ (e + 5.0)$.
Mt. Wilson, " 1,780 " $p_a = p_w - 0.0020\ (e + 2.0)$.
La Confianza (4,483), Mt. Whitney (4,420), Humahuaca (2,939), Maimará (2,384), have similar equations, but the series of observations is too short to determine the constants in $p_a = p_w + F\ (e + B)$.

Type II.

Bassour, height 1,160 meters, $p_a = p_w$.
Mt. Weather, " 526 " $p_a = p_w$.
There is some uncertainty about the equation for Mt. Weather on account of the indecisive data of I_o.

Type III.

Jujuy, height 1,302 meters, $p_a = p_w + 0.0060\ (e - 6.0)$.
Cordoba, " 438 " $p_a = p_w + 0.0087\ (e - 2.6)$.
Pilar, " 340 " $p_a = p_w + 0.0087\ (e - 2.6)$.
Potsdam, " 89 " $p_a = p_w + 0.0090\ (e + 0.0)$.
Washington " 34 " $p_a = p_w + 0.0100\ (e + 2.4)$.

The coefficient F diminishes from + 0.0100 at Washington

(34) through 0.000 at Bassour (1,160), to − 0.0080 at La Quiaca (3,465); the amount of the B is very variable with the local conditions of the vapor-pressure e at the station.

Apply these equations to the values of I_1 and compute I_a.

TABLE 91

WASHINGTON, D. C., STATION EQUATION, $p_a = p_w + 0.0100\ (e \times 2.4)$

No.	p_w	e	$e + 2.4$	A	p_a	I_1	I_a	I_{sea}	ΔI	
7	.884	3.78	6.2	.062	.944	1.410	1.497	The average value of I_a at sea level = 1.525	The correction from the station to sea level	
16	.874	3.00	5.4	.054	.928	1.402	1.511			
20	.865	2.09	4.5	.045	.910	1.409	1.548			
16	.854	2.60	5.0	.050	.904	1.392	1.540			
22	.846	3.34	5.7	.057	.903	1.386	1.535			
24	.834	4.06	6.5	.065	.899	1.410	1.568			
17	.824	3.80	6.2	.062	.886	1.353	1.527			
17	.815	3.69	6.1	.061	.876	1.418	1.619			
10	.804	4.30	6.7	.067	.871	1.334	1.532			mean 1.542
14	.795	5.02	7.4	.074	.869	1.349	1.553			
18	.784	5.10	7.5	.075	.859	1.316	1.532			
8	.773	5.10	7.5	.075	.848	1.303	1.537			
9	.764	5.19	7.6	.076	.840	1.335	1.590			
11	.756	6.10	8.5	.085	.841	1.272	1.512			
7	.743	7.03	9.4	.094	.837	1.260	1.506			
5	.736	7.69	10.1	.101	.837	1.262	1.508			
5	.714	8.85	11.3	.113	.827	1.252	1.514			
3	.703	7.40	9.8	.098	.801	1.170	1.461			
4	.696	6.06	8.5	.085	.781	1.212	1.552			mean 1.527
							1.534	1:525	−0.009	

The computed values of I_a from p_a, in place of I_o from p_w, plot near the dotted line in the diagram. The mean of the first nine values is 1.542, and of the last ten values 1.527, showing that the variations in p_w, and the effect of the vapor-pressure e, are quite well eliminated. A longer series would probably prove that the elimination is complete.

Table 92 contains the summary of the computed I_a for nine stations and their mean values. If these mean values of I_a are plotted, for the height z on the axis of ordinates and calories on the axis of abscissas, the *points fall nearly on a straight line, so that I_a diminishes in proportion to the height.* This line cuts the sea level at 1.525 calories, and the differences which corre-

TABLE 92

SUMMARY OF THE COMPUTED VALUES OF I_a.

Gr. Cal. / Cm.² Minute

Washington (84)	Potsdam (89)	Pilar (840)	Cordoba (438)	Mt. Weather (526)	Bassour (1160)	Jujuy (1302)	Mt. Wilson (1780)	La Quiaca (3465)	Maimará (2384)	Humahuaca (2939)	Mt. Whitney (4420)	La Confianza (4488)	Sea Level 0
1.497	1.540	1.613	1.559	1.527	1.627	1.748	1.742	1.997	Approximate mean values of I_a from incomplete series of observations				
1.511	1.561	1.501	1.580	1.593	1.680	1.707	1.767	2.006					
1.548	1.556	1.495	1.619	1.576	1.688	1.745	1.781	1.997					
1.540	1.525	1.573	1.588	1.643	1.711	1.693	1.783	2.011					
1.535	.....	1.521	1.612	1.660	1.672	1.717	1.797	2.012					
1.568	.....	1.554	1.644	1.599	1.673	1.768	1.743	.....					
1.527	.....	1.484	1.560	1.511	1.663	1.729	1.776	.....					
1.619	.....	1.657	1.576	1.534	1.698	1.747	.....	.....					
1.532	.....	.....	.....	1.630	1.683	1.763	.....	.....					
1.553	.....	.....	.....	1.553	1.645	1.654	.....	.....					
1.532	.....	.....	.....	1.643	1.686	.....	.....	.....					
1.537	.....	.....	.....	1.623	.....	.....	.....	.....					
1.590	.....	.....	.....	1.637	.....	.....	.....	.....					
1.512	.....	.....	.....	1.691	.....	.....	.....	.....					
1.506	.....	.....	.....	.....	.....	.....	.....	.....					
1.508	.....	.....	.....	.....	.....	.....	.....	.....					
1.514	.....	.....	.....	.....	.....	.....	.....	.....					
1.461	.....	.....	.....	.....	.....	.....	.....	.....					
1.552	.....	.....	.....	.....	.....	.....	.....	.....					
1.534	1.545	1.550	1.592	1.601	1.675	1.727	1.771	2.005	1.852	1.930	2.138	2.142	1.525
Adopted mean reduction to the sea level													
-.004	-.013	-.047	-.067	-.076	-.150	-.180	-.246	-.480	-.327	-.405	-.613	-.617	.000

spond to the station height are readily computed. It is equally easy to determine the normal value of I_a at any station, and several examples are added for stations having only short series.

The Annual Mean Variations

The station equations can be applied to the individual p_w—groups, or to individual observations, during different years, and their mean sea-level values are collected in Table 93; also, the annual means are taken for all stations which were observed during the same year. We have such means extending from 1903 to 1914, and they are plotted on the upper curve of Fig. 78, giving minima in 1903, 1907, 1912, 1914. The 1907 and 1912 minima are already well known. In the second curve of Fig. 78 is reproduced the adopted mean meteorological curve from Bulletin No. 1, Oficina Meteorológica Argentina, written in 1910, which contained a summary of data till 1910, *and a forecast from* 1910 *to* 1915. It should be noted that the forecast for 1911, 1912, 1913, was very well verified, as it has actually been in all other of the Argentine meteorological data. In 1914 the minimum, which the forecast placed in 1915, seems to have come in 1914, but this is very unexpected, because there is an interval of only two years following 1912, whereas the ordinary periodic interval is 3.75 years. Another forecast is added with maximum in 1917 and minimum in 1919. It is necessary to maintain suitable solar observations, in order to study the causes of such irregular fluctuations in the output of the solar radiation, and several solar physics observatories, adapted to meteorological purposes, are indispensable in the interests of long-range forecasts.

The observatories at Pilar and La Quiaca, Argentina, are well adapted to supplement the work of Washington and Mt. Wilson in the United States.

The General Summary

It has been shown that the mean values of I_a plot along a straight line with a given slope. On Fig. 76 is plotted $S = 3.95$ the solar constant; ΣJ_a the curve of the absorbed radiation from

TABLE 93

Summary of the Mean Annual Values of I_A

Station	s	ΔI_a	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912	1913	1914
La Quiaca	3465	-.480	.....	.....	.....	.....	.....	.....	.....	.....	.....	1.514	1.527	1.440
Mt. Wilson	1780	-.246	.....	.....	1.531	1.520	.....	1.534	1.548	1.516	1.523	1.512	.....	.....
Bassour	1160	-.150	.....	.....	.....	.....	.....	.....	.....	.....	1.529	1.515	.....	.....
Mt. Weather	526	-.076	.....	.....	.....	.....	.....	1.559	1.549	.....	1.528	1.507	.....	.....
Cordoba	438	-.067	.....	.....	.....	.....	.....	.....	.....	.....	.....	1.517	1.542	1.492
Pilar	340	-.047	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	1.488
Washington	34	-.004	1.514	1.528	1.529	(Abbot)		.....	.....	.....	.....	.....	.....	.....
			(Kimball)		1.521	1.540	1.500	1.537	1.534	1.582	.....	.....	.....	.....
Annual means	.....	.....	1.514	1.528	1.527	1.530	1.500	1.543	1.544	1.538	1.527	1.516	1.535	1.473

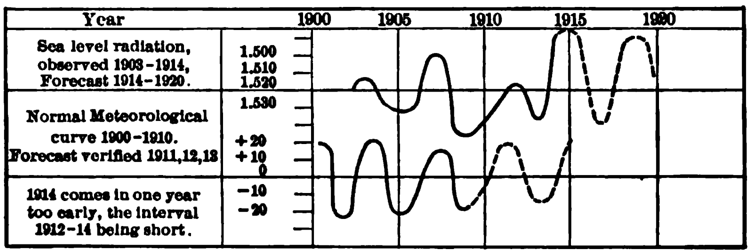


FIG. 78. Mean radiation intensity and long range forecast.

the outer limit to the station level, or it may be taken as $(S - \Sigma J_a) = B$, plotting ΣJ_a from the ordinate S; the density curve ρ is plotted with the abscissa as indicated; the pyrheliometer curve $I_a = A$ is plotted up to 4,500 meters, and its course follows the ρ curve so closely as to suggest that these two functions run out together, that is, at the solar constant 3,950 calories.

If Abbot's ordinates for the bolometer are filled up by proper interpolations for the several intermediate wave lengths, and the sums taken, we shall have good relative values of the several bolometer energy areas.

TABLE 94

SUMMARY OF THE BOLOMETER DATA

Bolometer Data	Abbot's Scale Sum.	Calories	Factor of Reduction	Calories B	Height z
Black body 6900° A......	80.30	3.876	20.7	3.876	23000
Abbot's extrapolated.....	67.26	.....	....	3.25	6500
Mt. Whitney............	61.38	.....	....	2.97	4420
Mt. Wilson.............	59.55	.....	....	2.88	1780
Washington, D. C........	51.19	.....	....	2.47	34

At the solar temperature 6900° A, for black body radiation, the value at the outer limit of the earth's atmosphere is 3.876, and corresponding to this the sum of Abbot's ordinates on his arbitrary scale is 80.30. The factor of reduction is taken 20.7, and the calories B for the several stations at the height z follow. Plot these on Fig. 76, and they seem to belong to the ΣJ_a curve. We conclude that the pyrheliometer data approximate the ρ density curve, and the bolometer data represent the selective radiation curve ΣJ_a. The total solar radiation now divides itself into three parts:

(1) Radiation intensity ΣJ_a used in producing temperature..1.46
(2) Radiation energy scattered back to space...............1.00
(3) Free radiation energy penetrating to the sea level.......1.52

Total or solar constant energy in calories..............3.98

The corresponding temperature would be 6930° A.

The I_0 – curve of the first method has been changed into the I_a – curve by the second method of computation.

GENERAL REMARKS

On Fig. 76 are drawn the typical curves from the data of Chapter VII: J_a the selective radiation absorbed in each 1,000-meter stratum, and ΣJ_a the total amount absorbed down to the given level; J_o the black body radiation in each 1,000-meter stratum, and ΣJ_o the total amount down to the given level; I_1 the computed radiation for the zenith sun on each level, I_o the computed I_1/p_w, and $I_a = I_1/p_a$; the density curve ρ; the solar constant $S = 3.950$, and Abbot's solar constant $E = 1.950$.

(1) Abbot multiplies I_o by a small bolometer factor, and adds a small function of e to produce $E = 1.950$ on each level. Bigelow's curve I_a, derived from the contact points for $p_w = 1.000$ at perfect transmission, crosses I_o at an angle, because Abbot's mean values of I_o are too large at low-level stations, and too small at high-level stations, both systems agreeing for $p_w = p_a$ at about 1,200 meters, the level of the cumulus cloud base. The pyrheliometer curve A, following closely upon the density curve ρ, will run out at about 3.950 colories, in conformity with the $\Sigma J_o = 3.950$ calories.

(2) The bolometer ordinates indicate an amount of energy well represented by the curve ΣJ_a or B, so that Abbot's small bolometer factor should be greatly increased, up to 2.00 in the higher levels. The theory which led to his small bolometer factor, namely, that the pyrheliometer registers the same amount of heat as that indicated by the bolometer, when the small corrections have been applied, is erroneous. The curve I_o does not represent the true pyrheliometer intensities, which are I_a, nor does it take any account of the energy scattered in the area between A and B, nor of the heat absorbed in making the temperatures T in the area between B and S.

(3) The pyrheliometer does not directly measure the solar constant at low levels, and the Bouguer formula is incapable of making the complete extrapolations; the bolometer does not measure the solar constant except by an approximate inference; the thermodynamics of the atmosphere affords data for com-

puting ΣJ_a and ΣJ_o in all levels, and the final sum for ΣJ_o at the sea level is the solar constant, about 3.950 calories.

(4) The thermodynamic computations are in accord with this interpretation of the pyrheliometric data and the bolometric data, in requiring about 3.950 calories to do the work actually measured in the atmosphere, and it would be impossible to maintain the existing temperature, pressure, and density distributions by expending only 1.950 calories. The pyrheliometer is useful in determining the relative annual variations of the solar radiation at any station; the bolometer is necessary in studying the relative absorptions of different wave lengths in the atmospheres of the sun and of the earth; but the thermodynamic data are required for the mutual interpretation of these two types of data.

This subject is treated at greater length in Bulletin No. 4, Oficina Meteorológica Argentina, 1914.

The Constants and Coefficients of Dry Air in the Kinetic Theory of Gases throughout the Atmosphere

It facilitates the discussion of the problems of the upper layers of the earth's atmosphere to have the data of the Kinetic Theory of Gases for Dry Air in a form for reference. Such a table of constants and coefficients for the (M. K. S.) and the (C. G. S.) systems is given in Table 95, and the results of their application from the bottom to the top of the earth's atmosphere as computed in Tables 96, 97. The fundamental data for $N = 6.062 \times 10^{23}$, $e = 4.774 \times 10^{-10}$, conform to the summary of constants by R. A. Millikan, *Physical Review*, August, 1913, the values for n, E_0, e, k, h, c_1, differing a little in consequence of slight variations in the adopted meteorological data for P, ρ, R. It is interesting to note that these fundamental physical quantities, and many others, can be computed, not only at the surface in standard conditions, but throughout the atmosphere. Similarly, the sun's data can be computed provided P, T, ρ, R, can be secured at the required points in the solar atmosphere. The formulas of Table 95 indicate the method of the computations, and numerous subordinate formulas can be derived from them. The transformation factors between the (M. K. S.) and the (C. G. S.) systems have been checked throughout this group of Formulas. In making transformations from (M. K. S.) to (C. G. S.) $M = 10^3$, $L = 10^2$; and from (C. G. S.) to (M. K. S.) $M = 10^{-3}$, $L = 10^{-2}$.

In Table 96, the kinetic energy per unit volume H, and U the inner energy per unit volume, decrease in a curve similar to those of P and K to vanishing values; q, the mean square velocity, and γ, the arithmetical mean velocity, diminish till they disappear at the outer limit; N, the number of molecules per kilogram-molecule, and n, the number of molecules per cubic meter, diminish slowly, but do not vanish. If the term N ought to remain constant, this diminution may go back to some additional physical forces not included in these simple formulas. Compare Fig. 70 for typical curves of the Dynamic, Thermodynamic, and the Kinetic Theory data, throughout the atmosphere.

A second more refined computation, with improved temperatures, has been undertaken. The data of Table 97 should be substituted for those of Table 96 above 40,000 meters, and the following remarks are based upon it. l_{max}, Maxwell's free path, increases from the surface upward, rapidly above 45,000 meters; ν, the number of collisions per second, diminishes with the height. The supposed height of the atmosphere, as derived from meteors at 100 kilometers, finds complete verification in these computations. m_H, the mass of the hydrogen atom, as computed, shows a small increase in value, probably due to some defects in our adopted simple formulas. Similarly, e_-, the elementary negative electric charge, assumed in this formula to hold a constant ratio to m_H, 3.4554×10^{-13} (M. K. S.), follows the fortunes of m_H.

TABLE 95

THE CONSTANTS AND COEFFICIENTS FOR DRY AIR IN THE (M. K. S.) AND (C. G. S.) SYSTEMS

Quantity		Formulas	M. K. S. Number	M. K. S. Log.	C. G. S. Number	C. G. S. Log.	Transf. Factor
Pressure		$P = \rho R T = \frac{1}{3} N. \rho u^2 = \frac{2}{3} n E_0$	101323.5	5.00571	1013235.	6.00571	$M L^{-1}$
Density and volume		$\rho = 1/v = n\, m\, m_H = n\, m/N$	1.29305	0.11162	.00129305	−3.11162	$M L^{-3}$
		$v = 1/\rho$	0.77336	−1.88838	773.36	2.88838	$M^{-1} L^3$
		$V = m\, v = m/\rho$	22.2235	1.34681	22223.5	4.34681	$M^{-1} L^3$
Temperature		T	273.	2.43616	273.	2.43616	1
Gas coefficient		$R = K/m$	287.0334	2.45793	2870334.	6.45793	L^2
		$K = m R = k N$	8248.2	3.91636	82482000.	7.91636	L^2
Mech. Equiv. Heat		A^1	4185.10	3.62171	41851000.	7.62171	L^2
		$K_A = K/A^1$	1.9708	0.29465	1.9708	0.29465	1
Specific Heat		C_p	993.5787	2.99720	9935787.	6.99720	L^2
		C_v	706.5453	2.84914	7065453.	6.84914	L^2
Molecular weight		m	28.736	1.45843	28.736	1.45843	1
Number of molecules	per M-mol.	$N = V n = m\, n\, v = K/k = 1/m_H$	6.062×10^{26}	26.78262	6.062×10^{23}	23.78262	M^{-1}
	per volume	$n = N/m\, v = H/E_0$	2.728×10^{25}	25.43581	2.728×10^{19}	19.43581	L^{-3}
Mean Kinetic Energy	For m	$P V = K T = \frac{1}{3} N m u^2 = \frac{2}{3} N E_0$	2.252×10^{6}	6.35252	2.252×10^{10}	10.35252	L^2
	For $m = 1$	$P v = R T = \frac{1}{3} N u^2 = \frac{2}{3} n\, v\, E_0$	7.836×10^{4}	4.89409	7.836×10^{8}	8.89409	L^2
Kinetic Energy as square velocity	one mol.	$u^2 = \frac{3 P v}{N} = \frac{3 R T}{N} = 2 v \frac{n}{N} E_0$	3.878×10^{-22}	−22.58859	3.878×10^{-15}	−15.58859	$M L^2$
	gram. mol.	$q^2 = N u^2 = 3 P v$	2.351×10^{5}	5.37121	2.351×10^{9}	9.37121	L^2
	Arithmet.	$\gamma^2 = \frac{8}{3\pi} q^2$	1.995×10^{5}	5.30003	1.995×10^{9}	9.30003	L^2

M. K. E. of one molecule.....	E_0	$= \frac{3}{2}\frac{P}{n} = \frac{1}{2} m u^2$	5.572×10^{-21}	-21.74599	5.572×10^{-14}	-14.74599	$M L^2$
Constant of Mol. Energy.....	ε	$= \frac{E_0}{T}$	2.041×10^{-23}	-23.30983	2.041×10^{-16}	-16.30983	$M L^2$
Boltzmann's Entropy Const...	k	$= \frac{2}{3}\varepsilon = K/N = R\, m\, m_H$	1.361×10^{-23}	-23.13374	1.361×10^{-16}	-16.13374	$M L^2$
Planck's Wirkungs quantum..	h	$= \frac{k^{4/3}}{c}\left(\frac{48\pi\, 1.0823}{7.39\times10^{-15}}\right)^{1/3}$	6.545×10^{-34}	-34.81590	6.54510^{-27}	-27.81590	$M L^2$
Wien-Planck constant in the..	c_2	$= \frac{c\,h}{k}$	1.44303×10^{-2}	-2.15928	1.44303	0.15928	L
Energy spectrum............	c_1	$= 8\pi c\, h$	4.935×10^{-24}	-24.69326	4.935×10^{-15}	-15.69326	$M L^3$
Kinetic Energy.............	H	$= \frac{3}{2} P = \frac{1}{2}\rho N u^2 = n E_0$	151986.	5.18180	1519860.	6.18180	$M L^{-1}$
Inner Energy...............	U	$= C v \rho T$	249412	5.39692	2494120.	6.39692	$M L^{-1}$
Velocity of light............	c		3×10^{8}	8.47712	3×10^{10}	10.47712	L
	c^2		9×10^{16}	16.95424	9×10^{20}	20.95424	L^2
Gravity acceleration.........	g_0		9.806	0.99149	980.6	2.99149	L
Viscosity..................	η		1.824^{-5}	-5.26102	1.824×10^{-4}	-4.26102	$M L^{-1}$
Free Path (Maxwell).........	l_{max}	$= \frac{\eta}{.30967\,\rho\,\gamma}$	1.020×10^{-7}	-7.00848	1.020×10^{-5}	-5.00848	L
Collisions, number per sec.....	ν	$= \gamma / l_{max}$	4.381×10^{9}	9.64154	4.381×10^{9}	9.64154	1
Diameter of molecule........	s	$= 6\sqrt{2}\, l_{max}\, 0.00203$	1.757×10^{-9}	-9.24465	1.757×10^{-7}	-7.24465	L
Elementary charge..........	e_-	$= 9646.8\, c/N$	4.774×10^{-15}	-15.67888	4.774×10^{-10}	-10.67888	$M L$
Mass hydrogen atom.........	m_H	$= \frac{\rho}{m\,n} = \frac{1}{9646.8\,c}\times e = 1/N$	1.6496×10^{-27}	-27.21738	1.6496×10^{-24}	-24.21738	M
Mass electric atom..........	m_-	$= m_H / 1865$	8.845×10^{-31}	−31 94670	8.845×10^{-28}	-28.94670	M
m_H/m_-..................	Constant		1865.	3.27068	1865.	3.27068	1
e_-/m_-..................	Constant		5.397×10^{15}	15.73218	5.397×10^{17}	17.73218	L
e_-/m_H..................	Constant		2.894×10^{12}	12.46150	2.894×10^{14}	14.46150	L

TABLE 96

THE DATA FOR THE KINETIC THEORY OF DRY AIR (M. K. S.) SYSTEM OF UNITS

From the Surface to 50,000 meters. Balloon Ascension, Huron, September 1, 1910

s Height, in Meters	H Kinetic Energy	U Inner Energy	q Mean Square Velocity	γ Arith. Mean Velocity	n Number per Cubic Meter	N Number per Kilogr. Molecule	l_{max} Freepath Length Maxwell	ν Number Collisions per Sec.	m_H Mass of Hydrogen Atom	e— Elementary Charge Electricity
50000	2.218×10^{-11}	3.632×10^{-11}	2.61	2.40	3.971×10^{11}	1.756×10^{22}	3.770×10^{5}	6.878×10^{-5}	5.698×10^{-23}	1.648×10^{-12}
49	6.276×10^{-6}	1.030×10^{-5}	16.00	14.75	1.418×10^{15}	6.605×10^{23}	8.152×10	1.809×10^{-3}	1.514×10^{-24}	4.881
48	2.222×10^{-2}	3.646×10^{-2}	52.10	48.00	3.987×10^{19}	7.000×10^{24}	7.496	6.403	1.428×10^{-25}	4.134×10^{-13}
47	1.076	1.765	91.32	84.13	1.981×10^{20}	2.147×10^{25}	2.710×10^{-3}	3.105×10^{4}	4.657×10^{-26}	1.348
46	9.845	16.15	125.64	115.76	1.767×10^{21}	4.071	4.079×10^{-4}	2.838×10^{5}	2.457	7.109×10^{-14}
45000	41.21	67.62	154.51	142.35	7.396	6.156	1.199	1.188×10^{6}	1.624	4.701
44	101.53	166.62	176.00	161.93	1.822×10^{22}	7.988	5.541×10^{-5}	2.927	1.252	3.622
43	176.26	289.55	190.63	175.63	3.167	9.371	3.453	5.086	1.067	3.088
42	255.28	418.90	201.07	185.25	4.582	1.043×10^{26}	2.518	7.858	9.591×10^{-27}	2.776
41	336.17	551.64	209.23	192.77	6.083	1.129	1.989	9.689	8.858	2.564
40000	418.93	687.50	215.99	199.00	7.519	1.208	1.648	1.207×10^{7}	8.812	2.406
39	505.87	830.16	221.95	204.49	9.079	1.270	1.403	1.458	7.872	2.278
38	600.03	984.63	226.97	209.60	1.077×10^{23}	1.335	1.212	1.729	7.493	2.168
37	704.85	1156.8	232.85	214.53	1.265	1.398	1.056	2.082	7.152	2.070
36	823.86	1351.8	238.16	219.42	1.479	1.463	9.240×10^{-6}	2.875	6.887	1.979
35000	960.17	1575.7	243.48	224.32	1.723	1.529	8.106	2.767	6.541	1.898
34	1121.9	1832.9	248.85	229.27	2.004	1.597	7.123	3.219	6.262	1.812
33	1298.7	2131.2	254.33	234.32	2.331	1.668	6.260	3.748	5.995	1.785
32	1508.0	2474.7	259.88	239.44	2.707	1.742	5.509	4.346	5.742	1.662
31	1749.6	2871.4	265.52	244.63	3.140	1.818	4.851	5.043	5.550	1.592

30000	2027.4	3327.1	271.23	249.89	3.639	1.897	4.276	5.343	5.271	1.525
29	2348.0	3853.3	277.04	255.25	4.214	1.979	3.772	6.767	5.052	1.462
28	2721.2	4465.6	283.01	260.75	4.884	2.065	3.244	8.089	4.842	1.401
27	3147.5	5164.9	289.03	266.29	5.649	2.154	2.935	9.072	4.642	1.343
26	3644.	5979.	295.21	271.93	6.540	2.247	2.590	1.050 × 10^8	4.450	1.288
25000	4225.	6933.	301.59	277.86	7.583	2.345	2.282	1.218	4.263	1.234
24	4900.	8041.	308.11	283.86	8.795	2.448	2.010	1.412	4.085	1.182
23	5690.	9337.	314.84	290.07	1.021 × 10^{24}	2.556	1.769	1.640	3.912	1.132
22	6616.	10857.	321.77	296.46	1.187	2.670	1.555	1.907	3.745	1.084
21	7695.	12627.	328.87	302.99	1.381	2.789	1.366	2.218	3.586	1.038
20000	8962.	14706.	336.27	309.74	1.608	2.915	1.199	2.583	3.431	9.929 × 10^{-15}
19	10458.	17162.	343.77	316.72	1.877	3.048	1.051	3.014	3.281	9.496
18	12197.	20016.	351.49	323.84	2.189	3.186	8.838 × 10^{-7}	3.516	3.139	9.086
17	14251.	23385.	359.48	331.26	2.558	3.332	8.063	4.108	3.001	8.684
16	16643.	27311.	367.63	338.70	2.987	3.485	7.061	4.797	2.869	8.304
15000	19479.	31965.	376.08	346.49	3.496	3.647	6.172	5.614	2.742	7.935
14	22775.	37378.	384.67	349.41	4.088	3.816	5.399	6.564	2.621	7.584
13	26614.	43672.	393.43	362.47	4.776	3.991	4.725	7.671	2.505	7.251
12	31096.	51028.	402.37	370.72	5.581	4.175	4.136	8.963	2.395	6.932
11	36226.	59446.	411.34	378.98	6.502	4.363	3.630	1.044 × 10^9	2.292	6.633
10000	42036.	68957.	420.28	387.21	7.544	4.555	3.196	1.212	2.195	6.354
9	48564.	79693.	429.14	395.37	8.716	4.749	2.825	1.400	2.106	6.236
8	55874.	91684.	437.92	403.46	1.008 × 10^{25}	4.945	2.505	1.610	2.002	5.852
7	64017.	105049.	446.61	411.47	1.149	5.143	2.230	1.845	1.845	5.626
6	73093.	119944.	455.23	419.41	1.312	5.344	1.991	2.107	1.871	5.415
5000	83132.	136416.	463.78	427.29	1.492	5.547	1.783	2.396	1.803	5.217
4	94168.	154521.	472.20	435.05	1.690	5.750	1.603	2.714	1.739	5.033
3000	106310.	174448.	480.56	442.74	1.908	5.955	1.445	3.135	1.679	4.860
2500	112920.	185300.	484.76	446.61	2.027	6.060	1.372	3.254	1.650	4.776
2000	119897.	196757.	488.97	450.51	2.157	6.180	1.304	3.456	1.618	4.693
1500	127300.	208900.	493.33	454.42	2.285	6.273	1.239	3.669	1.594	4.613
1000	135094.	221695.	497.49	458.34	2.426	6.382	1.177	3.894	1.567	4.535
500	143261.	235094.	501.72	462.24	2.571	6.491	1.119	4.129	1.541	4.458
392	145083.	238089.	502.64	463.10	2.604	6.515	1.107	4.182	1.535	4.442

TABLE 97

Extension of the Data for the Kinetic Theory of Gases

From 40,000 Meters to 90,000 (M. K. S.) System. Balloon Ascension, Uccle, November 9, 1911

z	H	U	q	γ	n	N	l_{max}	ν	m_H	e_-
Height. in Meters	Kinetic Energy	Inner Energy	Mean Square Velocity	Arith. Mean Velocity	Number per Cubic Meter	Number per Kilog. Molecule	Free Path Length Maxwell	Number Collisions per Sec.	Mass of Hydrogen Atom	Element Charge Electricity
90000	2.7689×10^{-13}	4.5510×10^{-13}	0.25409	0.23142	4.969×10^{2}	1.665×10^{20}	2.967×10^{-14}	7.799×10^{-14}	6.006×10^{-21}	1.738×10^{-9}
89	2.7485×10^{-12}	4.5178×10^{-12}	1.3389	1.2336	4.933×10^{9}	4.623×10^{21}	1.557×10^{-8}	7.922×10^{-9}	2.163×10^{-22}	6.260×10^{-10}
88	8.6760×10^{-11}	1.4260×10^{-10}	3.0743	2.8324	1.557×10^{10}	2.437×10^{22}	1.133×10^{-6}	2.501×10^{-6}	4.103×10^{-23}	1.187×10^{-10}
87	2.5159×10^{-10}	4.1275×10^{-9}	5.0041	4.6104	4.516×15^{11}	6.460×10^{22}	6.360×10^{-6}	7.250×10^{-6}	1.541×10^{-23}	4.480×10^{-11}
86	2.7456×10^{-8}	4.5040×10^{-8}	7.0672	6.5112	4.928×10^{12}	1.288×10^{23}	8.231×10^{-6}	7.731×10^{-4}	7.762×10^{-24}	2.246
85000	1.7536×10^{-7}	2.8773×10^{-7}	9.2378	8.5110	3.147×10^{13}	2.201	1.684	5.053×10^{-3}	3.609	1.044
84	7.9827	1.3097×10^{-6}	11.498	10.594	1.433×10^{14}	3.410	4.606×10^{-5}	2.300×10^{-2}	2.932	8.486×10^{-12}
83	2.8767×10^{-6}	4.7184	13.838	12.749	5.163	4.940	1.538	8.287	2.024	5.858
82	8.7348	1.4329×10^{-5}	16.246	14.968	1.568×10^{15}	6.809	5.947×10	2.517×10^{-1}	1.469	4.250
81	2.3274×10^{-5}	3.8177	18.717	17.245	4.177	9.087	2.572	6.706	1.106	3.202
80000	5.5945	9.1778	21.245	19.574	1.004×10^{16}	1.164×10^{24}	1.214	1.612×1	8.589×10^{-25}	2.485
79	1.2488×10^{-4}	2.0485×10^{-4}	23.858	21.980	2.241	1.468	6.109×1	3.598	6.811	1.971
78	2.6409	4.3323	26.584	24.492	4.740	1.823	3.219	7.609	5.485	1.588
77	5.3283	8.7408	29.420	27.105	9.563	2.233	1.766	1.535×10	4.479	1.296
76	1.0811×10^{-3}	1.6915×10^{-3}	31.993	29.817	1.851×10^{17}	2.640	1.004	2.971	3.787	1.096
75000	1.9234	3.1553	35.413	32.627	3.452	3.235	5.887×10^{-1}	5.542	3.091	8.946×10^{-13}
74	3.4723	5.6963	38.567	35.533	6.232	3.837	3.550	1.001×10^{2}	2.606	7.542
73	6.0841	9.9812	41.823	38.532	1.092×10^{18}	4.512	2.198	1.753	2.166	6.414
72	1.0380×10^{-2}	1.7029×10^{-2}	45.177	41.623	1.863	5.265	1.392	2.991	1.899	5.497
71	1.7281	2.8351	48.628	44.802	3.102	6.100	8.998×10^{-2}	4.979	1.639	4.744
70000	2.8148	4.6179	52.180	48.074	5.052	7.023	5.927	8.111	1.424	4.121
69	4.4914	7.3682	55.823	51.430	8.061	8.038	3.974	1.294×10^{3}	1.244	3.600
68	7.0847	1.1540×10^{-1}	59.560	54.874	1.263×10^{19}	9.151	2.707	2.027	1.098	3.982
67	1.0882×10^{-1}	1.7769	63.391	58.404	1.944	1.037×10^{25}	1.871	3.121	9.647×10^{-26}	2.792

66	1.6409	2.6918	67.310	62.014	2.945	1.169	1.812	4.728	8.556	2.476
65000	2.4499	4.0191	71.323	65.712	4.397	1.812	9.309 $\times 10^{-3}$	7.059	7.621	2.205
64	3.6071	5.9176	75.422	69.488	6.474	1.467	6.686	1.039 $\times 10^{4}$	6.815	1.972
63	5.2436	8.6024	79.610	73.347	9.411	1.635	4.854	1.511	6.117	1.770
62	7.5320	1.2357 ×1	83.886	77.286	1.352 $\times 10^{20}$	1.815	3.561	2.170	5.509	1.594
61	1.0696 ×1	1.7550	88.245	81.302	1.920	2.009	2.638	3.082	4.990	1.444
60000	1.5031	2.4661	92.690	85.396	2.698	2.216	1.972	4.331	4.513	1.306
59	2.0948	3.4368	97.242	89.590	3.760	2.439	1.484	6.036	4.100	1.187
58	2.9005	4.7585	101.92	93.903	5.206	2.680	1.124	8.358	3.732	1.080
57	3.9933	6.5518	106.74	98.342	7.167	2.939	8.546 $\times 10^{-4}$	1.151 $\times 10^{5}$	3.403	9.848 $\times 10^{-14}$
56	5.4625	8.9622	111.68	102.89	9.804	3.217	6.537	1.574	3.108	8.996
55000	7.4326	12.194	116.76	107.57	1.334 $\times 10^{21}$	3.516	5.023	2.142	2.844	8.230
54	10.059	16.504	121.98	112.38	1.805	3.838	3.877	2.899	2.606	7.541
53	13.535	22.210	127.32	117.31	2.430	4.182	3.007	3.901	2.391	6.921
52	18.136	29.754	132.82	122.37	3.255	4.550	2.342	5.226	2.198	6.360
51	24.166	39.648	138.43	127.55	4.337	4.944	1.832	6.964	2.023	5.854
50000	32.056	52.591	144.21	132.86	5.755	5.364	1.438	9.237	1.864	5.395
49	42.285	69.380	150.10	138.29	7.589	5.811	1.135	1.219 $\times 10^{6}$	1.721	4.980
48	55.394	90.890	156.06	143.79	9.942	6.282	9.008 $\times 10^{-5}$	1.596	1.592	4.607
47	72.125	118.34	162.13	149.37	1.295 $\times 10^{22}$	6.780	7.187	2.078	1.475	4.269
46	93.338	153.14	168.29	155.04	1.675	7.304	5.764	2.690	1.369	3.962
45000	102.12	197.07	174.53	160.79	2.156	7.856	4.645	3.461	1.273	3.770
44	153.59	251.99	180.83	166.61	2.757	8.435	3.764	4.426	1.186	3.431
43	194.89	319.75	187.16	172.24	3.498	9.035	3.074	5.603	1.107	3.203
42	245.12	402.16	193.46	178.24	4.399	9.654	2.523	7.063	1.036	2.998
41	305.14	500.64	199.68	183.97	5.476	1.028 $\times 10^{23}$	2.092	8.793	9.728 $\times 10^{-27}$	2.814
40000	375.06	615.34	205.73	189.54	6.731	1.092	1.714	1.081 $\times 10^{7}$	9.161	2.651

Summary of the Dimensions with Special Reference to the Equivalents

In transformations from the $(M.K.S.)$ to the $(C.G.S.)$ systems, the following dimensions should be observed:

Mass $= M = 10^3$. Length $= L = 10^2$. Time $= T = 1$ second.

Velocity $= L\,T^{-1}$. Momentum $= M.L\,T^{-1}$. Acceleration $= L\,T.^{-1}\,T.^{-1}$.

Force $= M.L\,T.^{-1}\,T^{-1} = M\,L\,T^{-2}$. Impulse $= M.L\,T^{-1}$.

Force per unit mass or force of acceleration . $= L\,T^{-2}$
Force per unit volume $= L\,T^{-2}.\,M\,L^{-3}$ . . $= M\,L^{-2}\,T^{-2}$

Specific Volume = Volume per unit mass . . $= M^{-1}\,L^3$
Specific Density = Mass per unit volume . . $= M\,L^{-3}$

Work = force × length $= M\,L\,T^{-2}.\,L$ . . $= M\,L^2\,T^{-2}$
Kinetic Energy = Mass times velocity squared $= M\,L^2\,T^{-2}$
Increase of kinetic or potential energy = work done . . . . $= M\,L^2\,T^{-2}$

Kinetic Energy per unit mass . . . . $= L^2\,T^{-2}$
Gravity potential = work per unit mass . . $= L^2\,T^{-2}$

Kinetic Energy per unit volume $= L^2\,T^{-2}.M\,L^{-3} = M\,L^{-1}\,T^{-2}$
Pressure = force per unit area $= M\,L\,T^{-2}.\,L^{-2} = M\,L^{-1}\,T^{-2}$
Pressure = work per unit volume $= M\,L^2\,T^{-2}.\,L^{-3} = M\,L^{-1}\,T^{-2}$

In the equations of Condition, we have for $T = 1$,

(a). $g\,(z_1 - z_0) = -\dfrac{P_1 - P_0}{\rho_{10}} - \tfrac{1}{2}\,(q_1^2 - q_0^2) - (Q_1 - Q_0)$, so that,

$$L.L = M\,L^{-1}.\,M^{-1}\,L^3 = L^2 = L^2 = 10^4 = 10{,}000.$$

(b). $\dfrac{Q_1 - Q_0}{v_1 - v_0} - P_{10} = \dfrac{U_1 - U_0}{v_1 - v_0} = K_{10} = c\,.\,T^a$, so that,

$$L^2.\,M\,L^{-3} = M\,L^{-1} = L^2.M\,L^{-3} = M\,L^{-1} = M\,L^{-1} = 10^3 \times 10^{-2} = 10.$$

Pressure has two definitions: (1) force per unit area, as Kilograms per square meter, or grams per square centimeter; (2) work per unit volume, as free heat per unit volume, hydrostatic pressure, inner energy per unit volume, radiation energy, and these all have the same dimension, $M L^{-1}$. Since the Erg and the Joule are units of work they must refer to the unit volume and not to the unit area.

$$\text{Hence } \frac{Joule}{volume} = M L^2 T^{-2}. L^{-3} = M L^{-1} T^{-2} = 10^3 \times 10^{-2} = 10,$$

$$\text{While } \frac{Joule}{area} = M L^2 T^{-2}. L^{-2} = M T^{-2} = 10^3 = 1{,}000$$

is distinctly erroneous.

Summary of Dimensions

In the conversion from $(M. K. S.)$ mechanical units into gr. cal./cm.^{2} minute, the correct factor becomes:

$$(M.K.S.) \times 10 \times 60/4.1851 \times 10^7 = 0.000014336$$

On the other hand, should the area L^2 be placed in the denominator under Joule, instead of the volume L^3, the factor $(M. K. S.) \times 1000 \times 60/4.1851 \times 10^7 = 0.0014336$ would be incorrect, inasmuch as the $\dfrac{Q - Q_0}{v_1 - v_0}$ is the work energy per unit volume and not per unit area.

The equations of condition assume two typical forms:

$$\text{(a)} \quad -(P_1 - P_0)\frac{60}{A} = g\,(z_1 - z_0)\,\rho_{10}\frac{60}{A},$$

$$M L^{-1} = L.L.\, M L^{-3} = M L^{-1} = 10$$

$$\text{(b)} \quad -\frac{(P_1 - P_0)}{\rho_{10}}\frac{60}{A} = g\,(z_1 - z_0)\frac{60}{A},$$

$$M L^{-1}.\, M^{-1} L^3 = L\,.\,L = L^2 = 10^4$$

For the equation (a), from $(M. K. S.)$ Mech. through $(C. G. S.)$ Mech., $(M. K. S.) \times 10 \times 60/4.1851 \times 10^7 = 0.000014336$

For the equation (a), from $(M. K. S.)$ Mech. through Kil. Cal./m.^{2} min., $(M. K. S.) \times 60/4.1851 \times 10^3 \times 10^3 = 0.000014336$

For the equation (b), from $(M. K. S.)$ Mech. through $(C. G. S.)$ Mech., $(M. K. S.) \times 10^4 \times 60/4.1851 \times 10^{10} = 0.000014336$

For the equation (b), from $(M. K. S.)$ Mech. through Kil. Cal./M^2 min., $(M. K. S.) \times 60/4.1851 \times 10^3 \times 10^3 = 0.000014336$

TABLE 98—SUMMARY OF DIMENSIONS

Formulas	Dimensions	M. K. S.	C. G. S.
Work	$M L^2 T^{-2}$	1 Joule $= \frac{\text{kilog.} m^2}{\text{sec.}^2} = 10^7$ ergs	1 erg $= \frac{\text{gr. cm.}^2}{\text{sec.}^2}$
$P = \rho R T$	$\frac{M L}{T^2} \cdot \frac{1}{L^3} = \frac{M}{L T^2}$	$\frac{\text{kilog.}}{m.\ \text{sec.}^2} = \frac{\text{Joule}}{m^3}$	$\frac{\text{gr.}}{\text{cm. sec.}^2} = \frac{\text{erg}}{\text{cm}^3}$.
ρ (m) = mol. wt.	$\frac{M}{L^3}$	$\frac{\text{kilog.}}{m^3}$	$\frac{\text{gr.}}{\text{cm.}^3}$
$U.$ $V = (m)\ n$ $(M) = M\ (m)$	$\frac{L^3}{M}$	$\frac{m^3}{\text{kilog.}}$	$\frac{\text{cm.}^3}{\text{gr.}}$
$R.\ C_p.\ C_v.$	$\frac{L^2}{T^2\ \text{deg.}}$	$\frac{m^2}{\text{sec.}^2\ \text{deg.}}$	$\frac{\text{cm.}^2}{\text{sec.}^2\ \text{deg.}}$
$K = R\ (m)$	$\frac{L^2\ (m)}{T^2\ \text{deg.}}$	$\frac{m^2 (m)}{\text{sec.}^2\ \text{deg.}}$	$\frac{\text{cm.}^2 (m)}{\text{sec.}^2\ \text{deg.}}$
A = mech. eq. H	$\frac{L M L}{T^2\ \text{cal}}$	$4.1851 \times 10^3\ \frac{\text{kilog.}\ m^2}{\text{sec.}^2}$	$4.1851 \times 10^{10}/\text{sec.}^2$
$W = \frac{\text{Watt}}{L^2 T}$	$\frac{(M)}{T^3\ \text{deg.}^4}$	$\frac{\text{kilog.}}{\text{sec.}^3\ \text{deg.}^4} = \frac{\text{Joule}}{m^2\ \text{sec. deg.}^4}$	$\frac{\text{1 Watt.}}{\text{cm.}^2\text{deg.}^4}$ $\frac{10^7\ \text{erg}}{\text{cm.}^2\text{sec. deg.}^4}$
$N = \frac{1}{m_H}$	$\frac{1}{M}$	$\frac{1}{\text{mass hydrogen atom}}$	$\frac{1}{\text{mass hydrogen atom}}$
n	$\frac{1}{L^3 (m)}$	$\frac{1}{m^3\ \text{(mol. wt.)}}$	$\frac{1}{\text{cm.}^3\ \text{(mol. wt.)}}$
$P V = K T$	$\frac{M L^{-1}}{T^2} \cdot \frac{(m) L^3}{M} = \frac{(m) L^2}{T^2}$	$\frac{\text{(mol. wt.)}\ m^2}{\text{sec.}^2}$	$(m) \frac{\text{cm.}^2}{\text{sec.}^2}$
$P v = R T$	$\frac{M L^{-1}}{T^2} \cdot \frac{L^3}{M} = \frac{L^2}{T^2}$	$\frac{m^2}{\text{sec.}^2}$.	$\frac{\text{cm}^2.}{\text{sec.}^2}$
$u^2 = \frac{3 P v}{N}$	$\frac{M L^2}{T^2}$	M-mol. $\frac{m^2}{\text{sec.}^2}$	M-mol. $\frac{\text{cm.}^2}{\text{sec.}^2}$
$q^2.\ \gamma^2.$	$\frac{L^2}{T^2}$	$\frac{m^2}{\text{sec.}^2}$	$\frac{\text{cm.}^2}{\text{sec.}^2}$
$E_v = \frac{3}{2} \frac{P}{n}$	$\frac{(M) L^2}{T^2}$	$M (m)\ \frac{m^2}{\text{sec.}^2}$	$M (m)\ \frac{\text{cm.}^2}{\text{sec.}^2}$
ϵ	$\frac{(M) L^2}{T^2\ \text{deg.}}$	$\frac{(M) m^2}{\text{sec.}^2\ \text{deg.}}$	$\frac{(M)\ \text{cm.}^2}{\text{sec.}^2\ \text{deg.}}$
$k = \frac{K}{N}$	$\frac{(M) L^2}{T^2\ \text{deg.}}$	$\frac{\text{kilog.}\ m^2}{\text{sec.}^2\ \text{deg.}} = \frac{\text{Joule}}{\text{deg.}}$	$\frac{\text{gr. cm.}^2}{\text{sec.}^2\ \text{deg.}} = \frac{\text{erg.}}{\text{deg.}}$
h	$\frac{M L^2}{T} = \frac{M L^2}{T^2} T$	$\frac{\text{kilog.}\ m^2}{\text{sec.}}$ = Joule sec.	$\frac{\text{gr. cm.}^2}{\text{sec.}}$ = erg sec.
$c_1 = 8 \pi ch$	$\frac{M L^3}{T^2}$	$\frac{\text{kilog.}\ m^3}{\text{sec.}^2}$ = Joule meter	$\frac{\text{gr. cm.}^3}{\text{sec.}^2}$ = erg cm.
$c_2 = ch/k$	L deg.	meter deg.	cm. deg.
$H = \frac{3}{2} P$	$\frac{M L}{T^2} \cdot L^{-2} = M L^{-1} T^{-2}$	$\frac{\text{kilog.}}{\text{met. sec.}^2} = \frac{\text{Joule}}{m^3}$	$\frac{\text{gr.}}{\text{cm. sec.}^2} = \frac{\text{erg}}{\text{cm.}^3}$
$U = C_v \rho T$	$\frac{L^2}{T^2\ \text{deg.}} \cdot \frac{M}{L^3} \text{deg.} = \frac{M}{L T^2}$	$\frac{\text{kilog.}}{m\ \text{sec.}^2} = \frac{\text{Joule}}{m^3}$	$\frac{\text{gr.}}{\text{cm. sec.}^2} = \frac{\text{erg}}{\text{cm.}^3}$
c	$L T^{-1}$	m/sec.	cm./sec.
g	$L T^{-2}$	$m/\text{sec.}^2$	$\text{cm.}/\text{sec.}^2$
$a = \frac{6 c_1 \alpha}{c_2^4}$	$\frac{M L^2}{T^2} \cdot \frac{1}{L^4 \text{deg.}^4} = \frac{M}{L T^2 \text{deg.}^4}$	$\frac{\text{kilog.}}{\text{m. sec.}^2\ \text{deg.}^4} = \frac{\text{Joule}}{m^3\ \text{deg.}^4}$	$\frac{\text{gr.}}{\text{cm. sec.}^2\ \text{deg.}^4} = \frac{\text{erg}}{\text{cm.}^3\ \text{deg.}^4}$
$\sigma = \frac{a c}{4}$	$\frac{M}{L T^2\ \text{deg.}^4} \cdot \frac{L}{T} = \frac{M}{T^3\ \text{deg.}^4}$	$\frac{\text{kilog.}}{\text{sec.}^3\ \text{deg.}^4} = \frac{\text{Joule}}{m.^2\ \text{sec. deg.}^4}$	$\frac{\text{gr.}}{\text{sec.}^3\ \text{deg.}^4} = \frac{\text{erg}}{\text{cm.}^2\ \text{sec. deg.}^4}$
$\sigma_A = \frac{\sigma}{A}$	$\frac{M}{T^3 \text{deg.}^4} \cdot \frac{T^2\ \text{cal.}}{L^2} = \frac{M\ \text{cal.}}{L^2 T\ \text{deg}^4}$	$\frac{\text{kilog. cal.}}{m^2\ \text{sec. deg.}^4}$	$\frac{\text{gr. cal.}}{\text{cm.}^2\ \text{sec. deg.}^4}$
$\sigma_w = \frac{\sigma}{W}$	$\frac{M}{T^3\ \text{deg.}^4}$	$\frac{\text{kilog.}}{\text{sec.}^3\ \text{deg.}^4} = \frac{\text{Joule}}{m^2\ \text{sec. deg.}^4}$	$\frac{\text{gr.}}{\text{sec.}^3\ \text{deg.}^4} = \frac{\text{erg}}{\text{cm.}^2\ \text{sec. deg.}^4}$

It should be noted that $a \times \frac{L}{T} = \sigma$, so that a is the work energy per unit volume. When this gains the velocity $L\,T_1$, it becomes σ or radiation through the unit area. The equation (*b*), page 414, $K_{10} = cT^a$ gives c of the same dimension as a, and it must be multiplied by a velocity to become comparable with σ in the Stefan Law. The computations for the radiation coefficients conform to these dimensions.

Similar computations have been successfully made for hydrogen and calcium vapor in the atmosphere of the sun, and the results are in conformity with the indications of the well-known observations. This knowledge of $P.\rho.R.T.$ in the superficial layers of the sun's envelope is of great value in interpreting the lines of the visible and radiation spectra. Indeed, the thermodynamic data are so complex that it will be impractical to determine them by direct observations. The change of R = constant to R = variable, that is, the transition from adiabatic to non-adiabatic conditions, converts the so-called constants k, h, c, c_2, a, σ into a series of coefficients which change from level to level. It follows that the laws of radiation for each chemical element are much more complicated than is implied in the use of the Stefan or Wein-Planck Laws with simple constants. This subject is of such far-reaching importance that it is reserved for further study and research.

General Problems in Atmospheric and Solar Physics

The purpose of the development of non-adiabatic meteorology has been to discover a method of computations applicable to all atmospheres, as those of the earth and the sun. The process consists in fixing an initial P_0, ρ_0, R_0, T_0 on some level z_0, and then, by assuming T_1 on the level z_1, determine the corresponding P_1, ρ_1, T_1 on that level, all to be checked by the gravity equation. From these values of T the entire system can be obtained. In the case of the sun the initial data can be fixed approximately as follows:

If the Boyle-Gay Lussac Law in the atmosphere of the earth is, $P\,U = R\,T$, this can be converted to the sun's atmosphere by multiplying each term with $28.028 = G/g = \gamma$, the ratio of the surface gravity on the sun to that on the earth, so that, $P\gamma \times U\gamma = R\gamma \times T\gamma$. The corresponding factor for all other terms follows very readily, and the entire thermodynamic system can be developed for each element by the method of trials. The check is found in the gravity equation.

$$g\,(z_1 - z_0) = -\frac{P_1 - P_0}{\rho_{10}} - (C\,p_a - C\,p_{10})\,(T_a - T_o).$$

The analogy between the physical condition in the atmosphere of the sun just above the photosphere, and in that of the earth between the levels 37,000 and 44,000 meters, is very interesting. Compare Fig. 79. In both there is apparently a very rapid change of temperature with the height, and a transition from excessively rarefied media to the tru egaseous condition. In the earth's atmosphere there is passage through such conditions as:

z in meters	37,000	44,000	
T (absolute)	211°	124°	M. K. S. System of units
P	437	98	
B (mercury in $M\,M$)	3.3	0.7	
ρ	0.025	0.0087	
R	82.77	91.49	

The physical transformations in this transition at 44,000 meters correspond with those sensitive states in the Geissler Tubes at which the electric and electro-magnetic phenomena are especially active. From this it is inferred that the origin of the auroral atmospheric electricity consists in the transformation of a portion of the incoming solar radiation of short wave lengths into ions, or free electric charges, and that these pursue their paths to the polar regions, as indicated in Fig. 69, generating the corresponding magnetic deflection vectors. In Table 95 the mass of the hydrogen atom is $m_H = \frac{1}{N} = 1.6496 \times 10^{-27}$, and

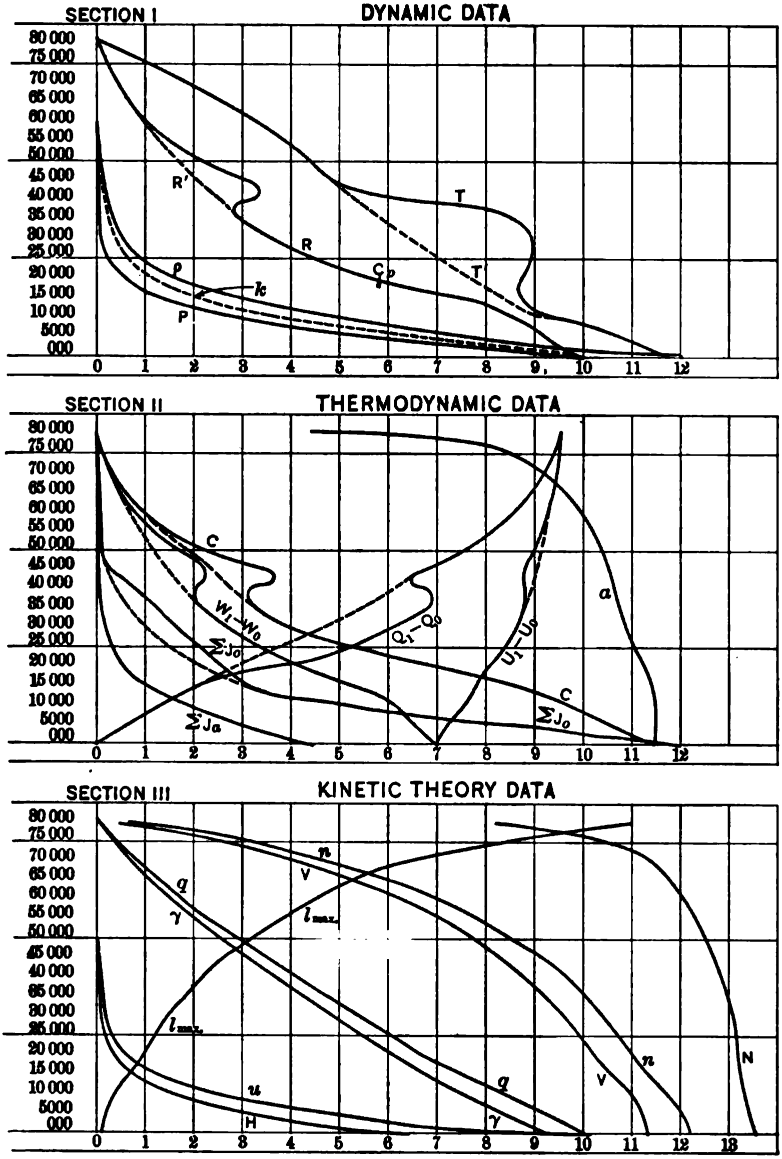

SECTION I
DYNAMIC DATA
R'
R
T
T'
ρ
k
P
SECTION II
THERMODYNAMIC DATA
C
a
SECTION III
KINETIC THEORY DATA
q
γ
n
V
u
H
N

Fig. 79.

if this is a constant, $N = 6.062 \times 10^{26}$ should, also, be a constant throughout the atmosphere. But by Tables 96, 97, N diminishes with the height, especially above 40 km.

z = 90,000	$N = 1.665 \times 10^{20}$
80,000	1.064×10^{24}
70,000	7.023×10^{24}
60,000	2.216×10^{25}
50,000	5.364×10^{25}
40,000	1.092×10^{25}
30,000	1.730×10^{26}
20,000	2.671×10^{26}
10,000	4.161×10^{26}
000	$N_0 = 6.147 \times 10^{26}$

The differences between N_0 and N may be taken as the number of transformed ions on the several levels, to be used in generating the auroral electric currents. *Furthermore, the region above* 44,000 *meters in the earth's atmosphere, with its excessively rarefied media, in electric sensitiveness under the impact of the incoming solar radiation of short wave lengths, is distinctly coronal in its physical conditions, and it corresponds with the deep corona that exists above the sun's chromosphere.* The rapid changes of temperature near the sun's photosphere produce a layer which is the source of great radiation energy outward, while the rapid temperature increase in the levels 44 to 37 km. of the earth's atmosphere is the evidence of an important absorption of radiation energy. Hence, the sharpness of the sun's visible disk is a phenomenon depending upon the thermodynamic conditions existing between the rapid decrease of temperature and the increase of radiation outward to space. The coronal and the photospheric regions in the atmospheres of the sun and the earth are, therefore, the physical counterparts of each other.

The laws of the absorption of the incoming radiation in the earth's atmosphere need further study. In place of interpreting the bolometer data in terms of the pyrheliometer data, and ascribing the great depletion of 4.00 gr. cal./cm.^{2} min. down to only 2.00 calories, as the result of a very imperfect action of the sun as a black body radiator, the result of the examination of the pyrheliometer and thermodynamic data seems to indicate that the pyrheliometer data should be interpreted in terms of the data of the bolometer, so that the "Solar constant" is about 4.00 calories, with the sun's effective radiation temperature 6950° *A*. In the Bouguer formula, the coefficients of transmission p_w as directly observed, and p_a as computed, have an interesting relation to the wave lengths in the spectrum. Abbot's values of p_w at different wave lengths were grouped together, at Mt. Wilson and at Washington, and diagrams of the mean values of p_w, as 0.936, 0.925, 0.915 . . . were plotted on the abscissas, 0.20 μ, 0.30 μ . . . 1.60 μ. If the values of p_w and p_a are plotted on these curves, they intersect at about wave length 0.57 μ. If 0.57 μ on $p_w = 1.00$ is taken as the maximum ordinate on the energy curve, this corresponds with about 5000° and 1.07 gr. cal./$cm.^2$ min. at the distance of the earth. Now, this is the average direct reading of the pyrheliometer at sea level, so that we infer that not only the aqueous vapor and dust, but also the density of the atmosphere, must be removed in discussing the coefficients of transmission. This conforms to the result shown in Fig. 76, where the second method of computation identifies the depletion of $I_a = I_1/p_a$ with the density ρ of the atmosphere. This research is not yet ready for final statement. It is very desirable that the excellent conditions prevailing at La Quiaca, at the elevation 3465 meters (11,037 feet) on the Bolivian Plateau, should be utilized for a solar physics observatory, equipped with a bolometer, spectroheliograph, and direct-vision spectroscope for prominences, to be operated in cooperation with Mt. Wilson and Washington, D. C. The conditions of living in La Quiaca are comparatively easy for observers, and there are complete railroad facilities for transportation from Buenos Aires. La Confianza, at 4483 meters (14,070 feet), can, also, be occupied the year around, as the village of San Vincente and the neighboring mining camp provide the necessary accommodations. It is important that the bolometric data should be obtained at these high levels, and there is an excellent opportunity for an expedition to make the necessary observations at those stations.

An inspection of the data of Chapter VII suggests that there are a very large number of problems in general physics that can be studied to advantage with the data which have been already developed. The laboratory is quite incapable of furnishing the fundamental relations that are shown to exist above the 45,000-meter level, where the pressure is a small fraction of one millimeter of mercury. Fortunately, the balloon ascensions to 30,000 meters cover a very interesting region, the so-called "isothermal layer," and they provide quite accurate observations of the temperature. The most important outstanding problem is to find a thermodynamic function for the kinetic energy of circulation, $-\frac{1}{2}(q_1^2 - q_0^2)$, in order to separate it from the free heat $-(Q_1 - Q_0)$.

It is hardly possible, in opening up so much new research material in the physics of the atmosphere, to have escaped imperfections and even errors, but it is thought desirable to indicate to meteorologists and astro-physicists some of the possible channels of investigation that appear to be accessible to such studies as are here illustrated.

INDEX

A

B

C

D

E

S

T